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Solution of One Dimensional Conformable fractional Heat Equation

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Abstract: ***The aim of this paper is to present the solution of one dimensional conformable fractional heat equation by applying conformable fractional derivative which is considered as more convenient definition of fractional derivative.***

Keywords: ***Caputo derivative, Conformable Fractional derivative, Riemann-Liouville definition.***

I. INTRODUCTION

Now a day's most popular area of research is fractional calculus. As fractional calculus is the generalization of classical theory of derivative, so many authors given various types of definitions of fractional derivative as well as fractional integral. R. Khalil [1] introduced conformable fractional derivative and fractional integral as a novel tool to solve different types of partial differential equations. This definition found to be most natural and also classical definition of first derivative is satisfied by taking $= 1$, along with some applications to fractional differential equations. Since there are so many definitions and theorems of fractional derivatives, one cannot easily decide which definition or theorem is useful for selected problem. Thabet Abdelwad [2] extended the definition of conformable fractional derivative to some basic concepts of classical theory like chain rule and also presented the Laplace transform. Mohammad R. S. R. [3] introduced a new kind of conformable fractional derivative on arbitrary time scales.

Yucel Ceneriz and et. al [4] defined Fourier transform based on conformable fractional derivative and space fractional heat equations are solved by using Fourier sine and cosine transform. Ajay Dixit and et. al [5] developed an analytical formulation to solve linear fractional order partial differential equations with given boundary conditions. Also discussed the method for the simultaneous fractional derivative in space as well as time. Ndolane Sene [6] proposed method to get the candidate solutions of the conformable differential equations. Mohammed K. A. Kaabar [7] formulated a two dimensional conformable fractional wave equation describing a circular membrane undergoing axisymmetric vibrations. Also obtained a solution by applying double Laplace transform method. Roshdi Khalil and et. al [8] introduced geometrical meaning of the conformable derivative by expressing the concepts of fractional cords and fractional orthogonal trajectories. M. Abu Hammad and et. al [9] have discussed the solution of the heat conformable fractional differential equation. Roshdi Khalil and et. al [10] presented some partial fractional differential equations in the form of conformable fractional derivative and solved same with the help of Fourier series and separation of variables. Also so many complicated valid illustrations of fractional derivative and integration which could not be applicable for each and every area of classical theory. Hence many researchers are always working for new definitions of fractional derivative. The novel definition of fractional derivative known as conformable fractional derivative is one of the best option to represent the concept on the expansion of fractional derivative. The history of research of fractional calculus begins with the most useful and familiar definitions.

1. Riemann-Liouville definition: If n is a positive integer and $\alpha \in [\, n-1, n\,)$, α derivative of f is given by

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx \qquad (1)$$

2. Caputo definition: If n is a positive integer and $\alpha \in [\, n-1, n\,, \alpha$ derivative of f is given by

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}} dx \qquad (2)$$

A novel definition of fractional derivative is presented by R. Khalil in [1] called "Conformable fractional derivative" which can be easily extends many concepts of classical theory and corresponding transforms.

3. Conformable fractional derivative definition:

Let $f: [\,0, \infty\,) \to R$ be a function α^{th} order conformable fractional derivative of f is defined by $T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, for\ all\ t > 0, \alpha \in (0,1)$. (3)

If f is α -differentiable in some $(0,a), a > 0$ and $\lim_{t\to 0^+} f^{\alpha}(t)$ exists, then define

$$f^{\alpha}(0) = \lim_{t\to 0^+} f^{\alpha}(t)$$

This new definition satisfies many properties which are given in the following theorem[9,10]

Theorem 1: Let $\alpha \in (0,1]$ and f , g be α-differentiable at a point t, then

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in R$
2. $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in R$
3. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$
4. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$
5. $T_\alpha(C) = 0$, Where $f(t) = C$ is a constant function.

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6. In addition g is differentiable then $T_\alpha(g)(t) = t^{1-\alpha}\frac{dg}{dt}$

Further

1. $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in R$
2. $T_\alpha(1) = 0$
3. $T_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}$, $c \in R$
4. $T_\alpha(sinbx) = bx^{1-\alpha}cosbx$, $b \in R$
5. $T_\alpha(cosbx) = -bx^{1-\alpha}sinbx$, $b \in R$
6. $T_\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1$

The left limit and right limit of fractional derivative is presented in [2] using conformable fractional derivative which is more analogues to classical theory defined as

Definition:

The left limit of fractional derivative from the point a of a function $g: [a,\infty)$ having order $0 < \alpha \le 1$ can be represented as

$$(T_\alpha^a g)(t) = \lim_{\epsilon \to 0} \frac{g(t + \varepsilon(t-a)^{1-\alpha}) - g(t)}{\varepsilon}$$

When $= 0$, we get T_α. If $(T_\alpha g)(t)$ exists on (a, b) therefore

$$(T_\alpha^a g)(a) = \lim_{t \to 0^+}(T_\alpha^a g)(t)$$

The right limit of fractional derivative having order $0 < \alpha \le 1$ terminating at b of g can be expressed as

$$({}_\alpha^b T g)(t) = -\lim_{\epsilon \to 0} \frac{g(t + \varepsilon(b-t)^{1-\alpha}) - g(t)}{\varepsilon}$$

If $({}^bT_\alpha g)(t)$ exists on (a, b) then

$$({}^bT_\alpha g)(b) = \lim_{t \to b^-} ({}^bT_\alpha g)(t)$$

II. Solution of Heat Equation

The one dimensional heat equation is given by

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < l,\ t > 0$$

With the conditions

$$T(0,t) = 0,\ t > 0$$

$$T(l,t) = 0,\ t \ge 0$$

$$T(x,0) = T_0,\ 0 \le t \le l$$

Which can be expressed in fractional calculus as

$$\frac{\partial^\alpha T}{\partial t^\alpha} = k\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < l,\ t > 0 \text{ and } 0 < \alpha \le 1 \qquad (4)$$

With the conditions

$$T(0,t) = 0, t > 0 \qquad (5)$$

$$T(l,t) = 0, t \ge 0 \qquad (6)$$

$$T(x,0) = T_0,\ 0 \le t \le l \qquad (7)$$

The above fractional partial differential equation can be converted into linear differential equations with constant coefficients with the help of variable separable form and then it can be solved

Let

$T = P(x)Q(t)$ be the required solution of above equation then

$$\frac{d^\alpha P(x)Q(t)}{dt^\alpha} = k\frac{d^2P(x)Q(t)}{dx^2} \qquad (8)$$

$$\frac{d^\alpha Q(t)}{dt^\alpha}\Big/ kQ(t) = \frac{d^2P(x)}{dx^2}\Big/ P(x) = \omega^2$$

$$\frac{d^\alpha Q(t)}{dt^\alpha} - k\omega^2 = 0 \quad \textbf{and} \qquad \textbf{(9)}$$

$$\frac{d^2P(x)}{dx^2} - k\omega^2 = 0 \qquad \textbf{(10)}$$

Solving above equations we get

$$Q = c_1 e^{-\omega^2\frac{k}{\alpha}t^\alpha} \qquad \textbf{(11)}$$

$$\frac{d^2P}{dx^2} - \omega^2 P = 0 \qquad \textbf{(12)}$$

$$P = c_2 cos\omega x + c_3 sin\omega x$$

Appling conditions (5) and (6), we get,

$$P = c_3 sin\left(\frac{n\pi x}{l}\right)$$

Combining these solutions, we get,

$$T = \sum_{n=1}^{n=\infty} c_1c_2\, sin\left(\frac{n\pi x}{l}\right) e^{-\omega^2\frac{k}{\alpha}t^\alpha} \qquad \textbf{(13)}$$

$$T = \sum_{n=1}^{n=\infty} b\, sin\left(\frac{n\pi x}{l}\right) e^{-\omega^2\frac{k}{\alpha}t^\alpha} \qquad \textbf{(14)}$$

By applying condition (7), we get

$$T_0 = \sum_{n=1}^{n=\infty} b\, sin\left(\frac{n\pi x}{l}\right) \qquad \textbf{(15)}$$

By using Fourier series,

$$b = \frac{2}{l}\int_0^l T_0\, sin\left(\frac{n\pi x}{l}\right)dx = \frac{4T_0}{\pi}, \qquad n = 2N+1$$

The temperature at any time t is given by

$$T = \frac{4T_0}{\pi}\sum_{N=1}^{N=\infty} sin\left(\frac{(2N+1)\pi x}{l}\right) e^{-\omega^2\frac{k}{\alpha}t^\alpha} \qquad \textbf{(16)}$$

III. CONCLUSION

In this paper one dimensional fractional heat equation is presented in the form of conformable fractional derivative and solved the same using the novel concept of fractional derivative known as conformable fractional derivative. Also it has been proved that conformable fractional derivative is more convenient to find analytical solution for partial fractional differential equations as compared to other definitions of fractional derivative.

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