Fuzzy Metric Dimension of Fuzzy Hypercube $Q_n$ and Fuzzy Boolean Graphs

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Abstract: Let $G = (V, E, \mu)$ be a fuzzy graph. Let $M$ be a subset of $V$. $M$ is said to be a fuzzy metric basis of $G$ if for every pair of vertices $x, y \in V - M$, there exists a vertex $w \in M$ such that $d(w, x) \neq d(w, y)$. The number of elements in $M$ is said to be fuzzy metric dimension (FMD) of $G$ and is denoted by $\beta(G)$. The elements in $M$ are called as source vertices. In this paper, we study the fuzzy metric dimension of fuzzy hypercube $Q_n$, fuzzy Boolean Graph $BG_2(G)$ and fuzzy Boolean Graph $BG_4(G)$.

Keywords: fuzzy Boolean graph $BG_2(G)$, fuzzy Boolean graph $BG_4(G)$, fuzzy hypercube $Q_n$, fuzzy metric dimension.

I. INTRODUCTION

A fuzzy graph $[7]$ $G$ is a 2-tuple $(V, E)$ where $V$ is a nonempty set of vertices $\{v_1, v_2, \ldots, v_n\}$ and $E$ is the nonempty finite set of edges such that $\mu: V \rightarrow [0, 1]$ and $\sigma: V \times V \rightarrow [0, 1]$ where $\sigma(v_i, v_j) \leq \min(\mu(v_i), \mu(v_j))$ for $i \neq j$.

Any for $v \in V$, if $\mu(v) = 0$ then we call $v$ as an active vertex. If $\mu(v) > 0$ then we call $v$ as an inactive vertex. We assume that all the vertices as active vertices. We use the notation $e_{ij}$ to denote the edge connecting the vertices $v_i$ and $v_j$. The weight of the edge $e_{ij}$ is given by $\sigma(v_i, v_j)$ and is denoted by $w(e_{ij})$.

A fuzzy path $[7]$ from a vertex $v_i$ to a vertex $v_j$ in a fuzzy graph is a sequence of distinct vertices and edges starting from $v_i$ and ending at $v_j$. This is denoted by $P(v_i, v_j) = P$.

If $v_i$ and $v_j$ coincide in a fuzzy path $P$ then we call this sequence as a fuzzy cycle. Let $P_{ij}$ be the set of all fuzzy paths $P$ from $v_i$ to $v_j$. For $v_i, v_j \in V$, we define the fuzzy set $\mu_{ij}: P_{ij} \rightarrow [0, 1]$ by $\mu_{ij}(P) = \min_{e_{ij} \in E} w(e_{ij})$ where $P \in P_{ij}$. Here $\mu_{ij}(P)$ is called the weight of the path $P$. The fuzzy path $P \in P_{ij}$ for which $\mu_{ij}(P)$ is minimum, is called as a fuzzy shortest path (FSP) between $v_i$ and $v_j$. The weight of this FSP is denoted by $d'(v_i, v_j)$. Thus, $d'$ can be viewed as a fuzzy set, $d': V \times V \rightarrow [0, 1]$ where $d'(v_i, v_j) = \min_{P \in P_{ij}} (\mu_{ij}(P))$ and $d'(v_i, v_i) = 0$.

For any two fuzzy shortest paths $P$ and $Q$ between $v_i$ and $v_j$, we consider the path with lesser number of intermediate vertices.

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In 1995, George and Veeramani defined the 3-tuple $(V, d^*, t)$ as $d^*(v_i, v_j, t) = \frac{t}{t + d'(v_i, v_j)}$, where $t$ is the number of intermediate vertices in the shortest path from which $d^*$ is calculated [5]. $N(v_i, v_j)$ is defined as the number of intermediate vertices between $v_i$ and $v_j$ in fuzzy shortest path (FSP) and $d^*(v_i, v_j, t)$ is denoted as $d^*(v_i, v_j, t)$.

Let $G = (V, E, \mu)$ be a fuzzy graph. Let $M$ be a subset of $V$. $M$ is said to be a fuzzy metric basis of $G$ if for every pair of vertices $x, y \in V - M$, there exists a vertex $w \in M$ such that $d(w, x) \neq d(w, y)$. The number of elements in $M$ is said to be fuzzy metric dimension (FMD) of $G$ and is denoted by $\beta(G)$. The elements in $M$ are called as source vertices. In 2012, Praba et al introduced and defined the fuzzy metric dimension of fuzzy graphs [7]. In 2016, Bhanumathi and Thusleem Furjana studied the fuzzy metric basis of some standard fuzzy graphs $G$, fuzzy metric basis of Total graph, middle graph and subdivision graph of some standard fuzzy graphs $G$ [1], [2]. Also they have determined the fuzzy metric basis of fuzzy Cartesian product of some fuzzy graphs [3]. In this paper, we determine some new bounds for the fuzzy metric dimension of fuzzy hypercube $Q_n$, $Q_2$, $Q_3$. Also, we study the fuzzy metric basis of fuzzy Boolean graph $BG_2(G)$ for some standard fuzzy graphs $G$ and fuzzy Boolean graph $BG_2(G)$ for some standard fuzzy graphs $G$.

Theorem 1.1 [5] $d'$ is a metric.

Theorem 1.2 [7] If $G$ is a path then $\beta(G) = 1$.

Theorem 1.3 [7] If $P_{ij}$ is a path on $n$ vertices and $v_k$ is an intermediate vertex in $P_{ij}$, $v_i$ and $v_j$ are two vertices on either side of $v_k$ then $\tilde{d}(v_k, v_i) = \tilde{d}(v_k, v_j)$ if and only if $N(v_k, v_i) = N(v_k, v_j)$, and $d'(v_k, v_i) = d'(v_k, v_j)$.

Theorem 1.4 [7] Let $P_{ij}$ be a path on $n$ vertices and $v_k$ is an intermediate vertex in $P_{ij}$. If $v_i$ and $v_j$ are two vertices on either side of $v_k$ such that $N(v_k, v_i) = N(v_k, v_j)$ then $\tilde{d}(v_k, v_i) = \tilde{d}(v_k, v_j)$ if and only if $d'(v_k, v_i) = d'(v_k, v_j)$.

Theorem 1.5 [7] If $C_n$ is fuzzy cycle then $\beta(C_n) \leq 2$.

Definition 1.6 A graph $G$ is said to be decomposable [4] into Hamiltonian cycles if its edge set can be partitioned into Hamiltonian cycles. A graph is said to admit cycle decomposition (respectively Hamiltonian decomposition) if its edge set can be partitioned into cycles (respectively Hamiltonian cycles).
Let $C_n$ denote the cycle of length $n \geq 3$. If $C_m$ and $C_n$ have vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively, we denote the vertices and edges of $C_m \times C_n$ by $\{u_i, v_j\}$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$, and $|E(C_m \times C_n)| = 2nm$. Thus, if $C_m \times C_n$ admits Hamiltonian decomposition then the number of cycles in such decomposition is two. Two Hamiltonian cycles in a graph are said to be edge-disjoint if there exists no common edges in them.

**Theorem 1.7** [8] The binary n-cube, with n even or equivalently the product of (n/2) cycles, $C_4 \times C_4 \times \cdots \times C_4$ can be partitioned into (n/2) Hamiltonian cycles.

**Definition 1.8** A star [6] in a fuzzy graph consist of two node sets $V$ and $U$ with $|V| = 1$ and $|U| > 1$, such that $\mu(v, u_i) > 0$ and $\mu(u_i, u_i) = 0$, $1 \leq i \leq n$. It is denoted by $S_{1,n}$.

## II. FUZZY METRIC DIMENSION OF FUZZY HYPERCUBE.

In this section we determine fuzzy metric basis of fuzzy Hypercube $Q_n$ for $n = 4$ and $n = 6$.

**Definition 2.1** The fuzzy hypercube or n-fuzzy cube $Q_n$ is the graph whose vertex set is the set of all n-dimensional Boolean vectors in which two vertices are joined if and only if they differ in exactly one coordinate.

### A. Fuzzy Metric Dimension of Hypercube $Q_n$.

**Theorem 2.1** If $G = Q_4$, then $2 \leq \beta(G) \leq 4$.

**Proof:** $G = Q_4 = K_2 \times K_2 \times K_2 \times K_2 = C_4 \times C_4$. Let $V_1 = \{u_1, u_2, u_3, u_4\}$ be the vertex set of one $C_4$ and $V_2 = \{v_1, v_2, v_3, v_4\}$ be the vertex set of the other $C_4$. Then $V(G) = V_1 \times V_2 = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4)\}$. $Q_4$ can be partitioned into two Hamiltonian fuzzy cycles as follows:

$C_1: (u_1, v_1)u_2v_4u_3v_3u_4v_2u_1v_1u_1\ v_2v_3u_4v_4u_3v_3u_2v_2u_1v_1u_1$

$C_2: (u_1, v_1)u_2v_4u_3v_3u_4v_2u_1v_1u_1v_2v_3u_4v_3u_3v_2u_2v_2u_1v_1u_1$

We will write $Q_4 = C_4 \times C_4$ as the union of two Hamiltonian fuzzy cycles, that is $C_4 \times C_4 = C_1 \cup C_2$

In $C_1$, take $u_1v_1$ as a source vertex, let $P_1$ be the path $u_1v_1u_2v_4u_3v_3u_4v_2u_1v_1u_1$, and $P_2$ be the path $u_1v_1u_2v_4u_3v_3u_4v_2u_1v_1u_1$. In $C_2$, take $u_1v_1$ as a source vertex. Let $P_1$ be the path $u_1v_1u_2v_4u_3v_3u_4v_2u_1v_1u_1$, and $P_2$ be the path $u_1v_1u_2v_4u_3v_3u_4v_2u_1v_1u_1$. Here we calculate the fuzzy metric dimension of $C_4 \times C_4$.

#### Case i:

In $C_1$, let $u_jv_j$ and $u_iv_i$ $(i = 1, 2, 3, 4)$ be two vertices on $C_1$ such that both $u_jv_j$ and $u_iv_i \in P_1$ or $P_2$ $(i = 1, 2, 3, 4)$ and $i = j \neq 1$ will be in same path. Thus, $\beta(C_1) = 1$. In $C_2$, let $u_jv_j$ and $u_iv_i$ $(i = 1, 2, 3, 4)$ be two vertices on $C_2$ such that both $u_jv_j$ and $u_iv_i \in P_1$ (or $P_2$) $(i = 1, 2, 3, 4)$ and $i = j \neq 1$ will be in same path. Thus, $\beta(C_2) = 1$. Thus, $\beta(C_1 \times C_2) = \beta(C_1) \cup \beta(C_2) = \{u_1v_1, u_1v_1\}$. Therefore, $\beta(Q_4) = 2$.

#### Case ii:

In $C_1$, if the two vertices $u_i\ v_i$ and $u_j\ v_j$ $(i = 1, 2, 3, 4$ and $i \neq j = 1)$ belongs to either $P_1$ (or $P_2$), then by case (i) we get, $\beta(C_1) = 1$. In $C_2$, if $u_i\ v_i$ and $u_j\ v_j$ $(i = 1, 2, 3, 4$ and $i \neq j = 1)$ such that the FSP for $u_i\ v_i$ from source vertex $u_1v_1$ is through $P_2$ and FSP for $u_j\ v_j$ from source vertex $u_1v_1$ is through $P_1$, then $\beta(C_1) = 1$. Hence, $\beta(C_2) = 2$ and $\beta(C_1 \times C_2) = \{u_1v_1, u_1v_1\}$. Therefore, $\beta(Q_4) = 2$.

Similarly, we get metric basis of $C_2$ as $\{u_1v_1, u_1v_1\}$ and $\beta(C_1 \times C_2) = \{u_1v_1, u_1v_1\}$. Hence, $\beta(Q_4) = 2$.
Include \( u_i, v_i \) as another source vertex so that \( N(u_i, v_i) \neq N(u_i, v_i) \) or \( N(u_i, v_i) \neq N(u_i, v_i) \). Continuing this process for all three paths in \( Q_n \), we get three source vertices for \( Q_4 \). \( \mathcal{M} = \{ u_i, v_i, u_i, v_i \} \). Hence, \( \tilde{\beta}(Q_4) \leq 3 \).

B. Fuzzy Metric Dimension of Hypercube \( Q_5 \)

**Theorem 2.3** If \( G = Q_n \), then \( \tilde{\beta}(G) \leq 4 \).

**Proof:** \( G = Q_2 \times Q_3 = C_4 \times C_4 \times C_4 \). If \( V_1 = \{ u_1, u_2, u_3, u_4 \} \) and \( V_2 = \{ v_1, v_2, v_3, v_4 \} \) such that \( V_1 \cap V_2 = \emptyset \), then \( V(G) = V_1 \times V_2 \) can be partitioned into four paths as follows:

\[
\begin{align*}
\mathcal{P}_1: & \quad u_1v_1x_1, u_1v_2x_1, u_1v_3x_1, u_1v_4x_1, u_1y_1x_1, u_1y_2x_1, u_1y_3x_1, u_1y_4x_1, \\
& \quad u_2v_1x_2, u_2v_2x_2, u_2v_3x_2, u_2v_4x_2, u_2y_1x_2, u_2y_2x_2, u_2y_3x_2, u_2y_4x_2, \\
& \quad u_3v_1x_3, u_3v_2x_3, u_3v_3x_3, u_3v_4x_3, u_3y_1x_3, u_3y_2x_3, u_3y_3x_3, u_3y_4x_3, \\
& \quad u_4v_1x_4, u_4v_2x_4, u_4v_3x_4, u_4v_4x_4, u_4y_1x_4, u_4y_2x_4, u_4y_3x_4, u_4y_4x_4.
\end{align*}
\]

In two paths \( P_1 \) and \( P_2 \), take \( u_1v_1x_1 \) as a source vertex. If two vertices \( u_iy_x \) or \( u_iy_x \) is through \( P_2 \), then \( \tilde{\beta}(Q_4) \leq 1 \). Include \( u_iy_x \) as another source vertex so that \( N(u_iy_x, u_iy_x) \neq N(u_iy_x, u_iy_x) \) or \( N(u_iy_x, u_iy_x) \neq N(u_iy_x, u_iy_x) \). Continuing this process for all four paths in \( Q_6 \), we get four source vertices for \( Q_6 \). \( \mathcal{M} = \{ u_i, v_i, u_i, v_i \} \). Hence, \( \tilde{\beta}(Q_6) \leq 4 \).

C. Fuzzy Metric Dimension of Hypercube \( Q_n \)

**Theorem 2.4** If \( G = Q_n \), then \( \frac{n}{2} \leq \tilde{\beta}(G) \leq n \).

**Proof:** \( Q_n \) can be decomposed into \( (n/2) \) Hamiltonian cycles, by Theorem 1.7. We get \( \frac{n}{2} \leq \tilde{\beta}(Q_n) \leq n \), by Theorem 1.5.

III. FUZZY METRIC DIMENSION OF FUZZY BOOLEAN GRAPHS \( BG_2(G) \) AND \( BG_2(G) \)

Let \( G(\sigma, \mu) \) be a fuzzy graph with its underlying set \( V \) and graph \( G^* = (\sigma^*, \mu) \). Let \( V(G) \) and \( E(G) \) be the vertex set and edge set of \( G^* \) respectively. The pair \( BG_2(G) = (\sigma_{BG_2(G)}, \mu_{BG_2(G)}) \) of \( G \) is defined as follows: Let the vertex set of \( BG_2(G) \) be \( V(G) \cup E(G) \). The fuzzy subset \( \sigma_{BG_2(G)} \) is defined on \( V(G) \cup E(G) \) as

\[
\sigma_{BG_2(G)}(u) = \sigma(u) \text{ if } u \in V(G) \quad \text{and} \quad \sigma_{BG_2(G)}(e) = \mu(e) \text{ if } e \in E(G)
\]
The fuzzy relation $\mu_{BG_2(G)}$ is defined as

$$\mu_{BG_2(G)}(u, v) = \mu(u, v)$$

if $u, v \in V(G)$ and $e = uv \in E(G)$

$$\mu_{BG_2(G)}(u, e) = 0$$

if $e = uv \notin E(G)$

$$\mu_{BG_2(G)}(e_i, e_j) = \mu(e_i) \land \mu(e_j)$$

if the edges $e_i$ and $e_j$ have no common incident vertex in $G$.

By the definition, $\mu_{BG_2(G)}(x, y) \leq \sigma_{BG_2(G)}(x) \land \sigma_{BG_2(G)}(y)$ for all $x, y \in V(G)$ and $E(G)$. Hence $\mu_{BG_2(G)}$ is a fuzzy relation on the fuzzy subset $\sigma_{BG_2(G)}$. Hence, the pair $BG_2(G)$: $(\sigma_{BG_2(G)}, \mu_{BG_2(G)})$ is a fuzzy graph and is termed as

**Boolean fuzzy graph $BG_2$** of $G$ - Second kind.

Similarly, the pair $BG_3(G)$: $(\sigma_{BG_3(G)}, \mu_{BG_3(G)})$ of $G$ is defined as follows. The fuzzy subset $\sigma_{BG_3(G)}$ is defined on $V(G) \cup E(G)$ as

$$\sigma_{BG_3(G)}(u) = \sigma(u)$$

for all $u \in V(G)$

$$\sigma_{BG_3(G)}(e) = \sigma(e)$$

for all $e \in E(G)$

The fuzzy relation $\mu_{BG_3(G)}$ is defined as

$$\mu_{BG_3(G)}(u, v) = 0$$

if $u, v \in V(G)$

$$\mu_{BG_3(G)}(u, e) = \mu(e)$$

if $e \in E(G)$ and $e$ is incident with $u$ in $G$.

$$\mu_{BG_3(G)}(e_i, e_j) = \mu(e_i) \land \mu(e_j)$$

if the edges $e_i$ and $e_j$ have no common incident vertex in $G$.

By the definition, $\mu_{BG_3(G)}(x, y) \leq \sigma_{BG_3(G)}(x) \land \sigma_{BG_3(G)}(y)$ for all $x, y \in V(G)$ and $E(G)$. Hence $\mu_{BG_3(G)}$ is a fuzzy relation on the fuzzy subset $\sigma_{BG_3(G)}$. Hence, the pair $BG_3(G)$: $(\sigma_{BG_3(G)}, \mu_{BG_3(G)})$ is a fuzzy graph and is termed as

**Boolean fuzzy graph $BG_3$** of $G$ - Third Kind.

In this section, we determine fuzzy metric basis of Fuzzy Boolean Graph $BG_2(G)$ for some standard graphs of $G$.

**Fuzzy Metric Dimension of $BG_2(G)$**

**Theorem:** If $G = BG_2(P_n)$ ($n > 3$), then $\tilde{P}(G) \leq \left\lfloor \frac{n + 2}{2} \right\rfloor$, when $n$ is odd.

$$\left\lfloor \frac{n + 3}{2} \right\rfloor$$

when $n$ is even.

**Proof:** Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $P_n$ and let $v_1v_2 = e_{12}$, $v_2v_3 = e_{23}$, ..., $v_{n-1}v_n = e_{n-1,n}$ be the edges of $P_n$. We denote $e_1 = e_1$, $e_2 = e_2$, ..., $e_{n-1} = e_n$. Edges of $BG_2(P_n)$ can be decomposed into $P_{n+1}$, $P_{2n+1}$.

**Case i:** $n$ is odd

Edges of $BG_2(P_n)$ can be decomposed into $(n/2)+1$ fuzzy paths as follows:

$$P_{1} = v_0, v_1, v_2, v_3, \ldots, v_{n-1}, v_n$$

$$P_{2} = e_1, e_2, e_3, \ldots, e_{(n+1)/2}, e_{(n+1)/2}$$

$$P_{3} = e_{(n+1)/2}, e_{(n+1)/2}, e_{(n+1)/2}, e_{(n+1)/2}, e_{(n+1)/2}$$

$$\ldots$$

$$P_{n+2} = e_{(n-2)/2}, e_{(n+1)/2}, e_{(n+1)/2}, e_{(n+1)/2}, e_{(n+1)/2}, e_{(n+1)/2}, \ldots, e_{(n-1)}$$

$$P_{2n+1} = v_0, v_1, v_2, v_3, \ldots, v_{n-1}, v_n$$

$$e_{n-2}, \ldots, e_n.$$
In two paths \( P_1 \) and \( P_2 \), take \( e_{n_1/2} \) as a source vertex. If two vertices \( v_k \) or \( e_k \in P_1 \) and \( v_l \) or \( e_l \in P_2 \) such that fuzzy shortest path from source vertex \( e_{n_1/2} \) for \( v_k \) or \( e_k \) is through \( P_1 \) and fuzzy shortest path from source vertex \( e_{n_1/2} \) for \( v_l \) or \( e_l \) is through \( P_1 \), then \( d(e_{n_1/2}, v_k) = \bar{d}(e_{n_2/2}, e_l) \) or \( d(e_{n_1/2}, e_l) = \bar{d}(e_{n_2/2}, v_k) \) if and only if \( N(e_{n_1/2}, v_k) = \bar{N}(e_{n_2/2}, e_l) \) or \( N(e_{n_1/2}, e_l) = \bar{N}(e_{n_2/2}, v_k) \). This implies that, \( \beta(BG_2(P_1 \cup P_2)) \neq 1 \). Include \( e_{n+2} \) as another source vertex so that \( N(e_{n+2}, v_k) \neq N(e_{n+2}, e_l) \) or \( N(e_{n+2}, e_l) \neq N(e_{n+2}, v_k) \).

Continuing this process for all \((n/2)+1 \) paths in \( BG_2(P_n) \), we get \((n/2)+1 \) source vertices for \( BG_2(P_n) \). \( \hat{M} = \{e_{n+2}, e_{n+4}, \ldots, e_{n+2n} \} \).

Hence, \( \beta(BG_2(P_n)) \leq (n+2)/2 \).

**Case ii**: \( n \) is even

Edges of \( BG_2(P_n) \) can be decomposed into \((n+1)/2)+1 \) fuzzy paths as follows:

\[ P_1: v_1 v_3 v_5 v_7 v_9 v_{11} v_{13} v_{15} v_{17} v_{19} v_{21} \]

\[ P_2: e_1 e_3 e_5 e_7 e_9 e_{11} e_{13} e_{15} e_{17} \]

\[ P_3: e_2 e_4 e_6 e_8 e_{10} e_{12} e_{14} e_{16} \]

\[ \ldots \]

\[ P_{(n+1)/2+1}: v_{n/2} v_{n/2-1} v_{n/2-2} v_{n/2-3} \ldots \ldots v_{n+1} e_{n+1} e_{n+3} \ldots \ldots e_{n+2n/2-2} e_{n+2n/2-1} e_{n+2n/2} \]

which has the same characterization as mentioned in the previous case.

Therefore, \( \hat{M} = \{e_{n+3/2}, e_{n+5/2}, e_{n+7/2}, \ldots, e_{n+2n} \} \).

Hence, \( \beta(BG_2(P_n)) \leq (n+3)/2 \).

**Fuzzy Metric Dimension of \( BG_2(C_n) \):**

**Theorem 3.2** If \( G = BG_2(C_n) \), then \( \beta(BG_2(C_n)) \leq \frac{n+1}{2} \), when \( n \) is odd.

**Proof:** Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices of \( C_n \) and let \( v_1 v_2 = e_{12}, v_2 v_3 = e_{23}, \ldots, v_{n-1} v_n = e_{n-1n} \). \( V_1 V_n = \in \) be the edges of \( C_n \).

**Case i:** \( n \) is even

Edges of \( BG_2(C_n) \) can be decomposed into \( n/2+2 \) fuzzy paths as follows:

\[ P_1: v_1 v_2 e_{23} e_{45} e_{67} e_{89} e_{11} e_{12} \ldots \ldots e_{n/2} v_{n/2} v_{n/2-1} v_{n/2-2} v_{n/2-3} \ldots \ldots e_{n/2-1} \]

\[ P_2: v_2 v_3 e_{23} e_{45} e_{67} e_{89} e_{11} e_{12} \ldots \ldots e_{n/2-1} v_{n/2} v_{n/2-1} v_{n/2-2} \ldots \ldots e_{1} \]

\[ P_3: v_3 v_4 e_{34} e_{56} e_{78} e_{91} e_{2} \ldots \ldots e_{n/2-2} v_{n/2-1} v_{n/2-2} v_{n/2-3} \ldots \ldots e_{2} \]

\[ \ldots \]

\[ P_{(n/2)+1}: v_{n/2} v_{n/2-1} v_{n/2-2} v_{n/2-3} \ldots \ldots v_{n+1} e_{n+1} e_{n+3} \ldots \ldots e_{n+2n/2-2} e_{n+2n/2-1} e_{n+2n/2} \]

which has the same characterization as mentioned in the previous case.

Therefore, \( \hat{M} = \{v_1, v_2, v_3, \ldots, v_{n/2}, v_{n/2+1}, v_{n/2+2} \} \).

Hence, \( \beta(BG_2(C_n)) \leq (n+3)/2 \).

**Fuzzy Metric Dimension of \( BG_2(nK_2) \):**

**Theorem 3.3** If \( G = BG_2(nK_2) \) then \( \beta(BG_2(G)) \leq n \).

**Proof:** Let \( v_1, v_2, v_3, \ldots, v_{2n} \) be the vertices of \( nK_2 \) and let \( v_1 e_{12}, v_2 e_{23}, \ldots, v_{2n-1} e_{2n-12n-1} \) be the edges of \( nK_2 \). We denote \( e_1 = e_2 = e_3 = \ldots = e_{2n-1} = e_n \). Edges of \( BG_2(nK_2) \) can be decomposed into \( K_n \) and \( n \) triangles.

**Case i:** \( n \) is even
We know that $K_n (n \geq 4)$ is decomposable into two fuzzy paths as follows:

(i) $n/2$ Hamiltonian fuzzy paths of length $n - 1$ (or)

(ii) $n - 1$ fuzzy paths of length $n/2$.

Thus, Edges of $BG_{(2nK_2)}$ can be decomposed into $n$ fuzzy paths as follows:

$P_1: v_{2i-1} v_{2i} e_2 c_2 e_2 c_2 e_{n-2} e_3 \ldots c_{(n+4)/2} e_{(n+4)/2} v_{2m-1}$  
$P_2: v_{2i-1} v_{2i} e_2 c_2 e_2 e_2 e_{n-2} e_3 \ldots c_{(n+6)/2} e_{(n+6)/2} v_{2m-1}$  
$P_3: v_{2i-1} v_{2i} e_3 c_2 e_2 c_2 e_2 c_2 e_{n-2} e_3 \ldots c_{(n+4)/2} e_{(n+4)/2} v_{2m-1}$  
$P_4: v_{2i-1} v_{2i} e_3 c_2 e_2 e_3 e_2 e_2 c_3 e_3 \ldots c_{(n+10)/2} e_{(n+10)/2} v_{2m-1}$  

$P_5: v_3 v_4$.  

$P_{2n}^2: v_{n-1} v_n$.

$P_{2n+1}^2: v_1 v_2 v_3$.  

$P_{2n+2}^3: v_3 v_4 v_5$.  

$P_{2n+3}^3: v_5 v_6 v_7$.  

$P_{2n}^n: v_{n-1} e_n v_n$.

In two paths $P_1$ and $P_2$, $v_1$ is fixed as a source vertex. If two vertices $v_1$ or $e_1 \in P_1$ and $v_1$ or $e_1 \in P_2$ such that fuzzy shortest path from $v_1$ for $v_k$ or $e_k$ is through $P_2$ and fuzzy shortest path from $v_1$ for $v_k$ or $e_k$ is through $P_1$, then $d (v_1, v_k) = d (v_1, v_k)$ or $d (v_1, v_k) = d (v_1, v_k)$ and $N(v_1, e_k) = N(v_1, e_k)$.

Include $v_1$ as another source vertex so that $N(v_1, v_k) = N(v_1, e_k)$ and $N(v_1, e_k) = N(v_1, v_k)$. This implies that, $\beta (BG_{(2nK_2)}) \neq 1$.

Include $v_1$ as another source vertex so that $N(v_1, v_k) = N(v_1, e_k)$ and $N(v_1, e_k) = N(v_1, v_k)$. Continuing this process for all $n$ paths in $BG_{(2nK_2)}$, we get $n$ source vertices for $BG_{(2nK_2)}$.

Therefore, $M = \{ v_1, v_3, \ldots , v_{2n-1}, e_1, e_3, \ldots , e_{2n-1} \}$. Hence, $\beta (BG_{(2nK_2)}) \leq n$.

Case ii: $n$ is odd.

Edges of $BG_{(2n+1)}$ can be decomposed into $n$ fuzzy paths of length three as follows:

$P_1: v_1 e_1 v_2$.  

$P_2: v_2 e_2 v_3$.  

$P_3: v_3 e_3 v_4$.  

$P_{2n}^n: v_6 e_6 v_7$.

In two paths $P_1$ and $P_2$, $e_1$ is fixed as a source vertex. If two vertices $v_1$ or $e_1 \in P_1$ and $v_1$ or $e_1 \in P_2$ such that fuzzy shortest path from $e_1$ for $v_k$ or $e_k$ is through $P_2$ and fuzzy shortest path from $e_1$ for $v_k$ or $e_k$ is through $P_1$, then $d (e_1, v_k) = d (e_1, v_k)$ or $d (e_1, v_k) = d (e_1, v_k)$.

Include $e_1$ as another source vertex so that $N(e_1, v_k) = N(e_1, e_k)$ and $N(e_1, e_k) = N(e_1, v_k)$. This implies that, $\beta (BG_{(2n+1)}) \neq 1$.

Include $e_1$ as another source vertex so that $N(e_1, v_k) = N(e_1, e_k)$ and $N(e_1, e_k) = N(e_1, v_k)$. Continuing this process for all $n$ paths in $BG_{(2n+1)}$, we get $n$ source vertices for $BG_{(2n+1)}$.

Therefore, $M = \{ e_1, e_3, \ldots , e_{2n-1}, v_1 \}$. Hence, $\beta (BG_{(2n+1)}) \leq n$.

B. Fuzzy Metric dimension of fuzzy Boolean graph $BG_{(G)}$.

In this section, We determine the fuzzy Metric basis of fuzzy Boolean graph $BG_{(G)}$ for some standard fuzzy graphs $G$.

Fuzzy Metric Dimension of $BG_{(P_n)}$.

Theorem: 3.5 If $G = BG_{(P_n)} (n > 3)$, then $\beta (G) \leq ...$
Edges of \(BG_{i}(C_n)\) can be decomposed into \(\frac{n}{2} + 3\) fuzzy paths as follows:

- For even \(n\): \(n/2 + 3\) paths.
- For odd \(n\): \(n/2 + 5\) paths.

**Proof:** Let \(v_1, v_2, v_3, \ldots, v_n\) be the vertices of \(C_n\) and let \(v_1v_2 = e_{12}, v_2v_3 = e_{23}, \ldots, v_{n-1}v_n = e_{n-1,n}, v_1v_n = e_{1n}\) be the edges of \(C_n\).

**Case i:** \(n\) is even

**Case ii:** \(n\) is odd

Fuzzy Metric Dimension of \(BG_i(C_n)\).

**Theorem:** If \(G = BG_i(C_n)\), then \(\hat{\beta}(BG_i(C_n)) \leq \left\lfloor \frac{n+5}{2} \right\rfloor\) when \(n\) is odd.

\(\left\lfloor \frac{n+6}{2} \right\rfloor\), when \(n\) is even.

**Proof:** Let \(v_1, v_2, v_3, \ldots, v_n\) be the vertices of \(C_n\) and let \(v_1v_2 = e_{12}, v_2v_3 = e_{23}, \ldots, v_{n-1}v_n = e_{n-1,n}, v_1v_n = e_{1n}\) be the edges of \(C_n\).

**Case i:** \(n\) is even

**Case ii:** \(n\) is odd

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Fuzzy Metric Dimension of Fuzzy Hypercube $Q_n$ and Fuzzy Boolean Graphs

**Theorem 3.7** If $G = BG_3(nK_2)$ then $\tilde{\beta}(BG_3(G)) \leq n$.

**Proof:** Let $v_1, v_2, v_3, \ldots, v_{2n}$ be the vertices of $nK_2$ and let $v_1v_2 = e_{12}, v_3v_4 = e_{34}, \ldots, v_{2n-1}v_{2n} = e_{2n-1,2n}$ be the edges of $nK_2$. We denote $e_{12} = e_1, e_{34} = e_2, \ldots, e_{2n-1,2n} = e_n$.

Edges of $BG_3(nK_2)$ can be decomposed into $K_n$ and $n$ paths of length two.

**Case i:** $n$ is even

We know that $K_n$ is decomposable into two fuzzy paths as follows:

(i) $n/2$ Hamiltonian fuzzy paths of length $n - 1$.

(ii) $n - 1$ fuzzy paths of length $n/2$.

Thus, Edges of $BG_3(nK_2)$ can be decomposed into $n$ fuzzy paths as follows:

$P_{1}: v_2, e_1, e_2, e_3, e_4, e_2, e_5, \ldots, e_{(n/4)}e_{(n/4)+2}v_{2m-1}$.

$P_{2}: v_2, e_1, e_2, e_3, v_5, e_6, e_7, \ldots, e_{(n/4)}e_{(n/4)+2}v_{2m-1}$.

$P_{3}: v_2, e_1, e_2, e_3, e_4, e_5, e_6, e_7, \ldots, e_{4m+2}e_{4m+4}v_{2m-1}$.

$P_{4}: v_2, e_1, e_2, e_3, e_4, e_5, e_6, e_7, \ldots, e_{(n/2)}e_{(n/2)+2}v_{2m-1}$.

In two paths $P_{1}$ and $P_{2}$, of $BG_3(G)$, take $e_1$ as a source vertex. If the two vertices $v_1$ or $v_1 \in P_{1}$ and $v_1 \in v_1 \in P_{2}$ such that fuzzy shortest path from $e_1$ to $v_1$ or $v_1$ is through $P_{2}$ and fuzzy shortest path from $v_1$ or $v_1$ is through $P_{1}$, then $d (v_1, v_1) = d (v_1, v_1)$ or $d (v_1, v_1) = d (v_1, v_1)$.

Continuing this process for all $n$ paths in $BG_3(nK_2)$, we get $n$ source vertices for $BG_3(nK_2)$. Hence, $\tilde{\beta}(BG_3(nK_2)) \leq n$.

**Case ii:** $n$ is odd

We know that $K_n$ is decomposable into $n$ fuzzy paths of length $(n-1)/2$.

The fuzzy metric dimension of $BG_3(S_{1,n})$ is as follows:

$P_{1}: v_1, e_1, e_2, e_1, e_2, \ldots, e_{(n-1)/2}v_{(n-1)/2}$.

$P_{2}: v_2, e_1, e_2, e_1, e_2, \ldots, e_{(n-1)/2}v_{(n-1)/2}$.

Hence, $\tilde{\beta}(BG_3(S_{1,n})) \leq n$. 

**Fuzzy Metric Dimension of $BG_3(S_{1,n})$**

**Theorem 3.8** If $G = BG_3(S_{1,n})$ then $\tilde{\beta}(BG_3(G)) \leq n$.

**Proof:** Let $v_{1}, v_{2}, v_{3}, \ldots, v_{2n}$ be the vertices of $S_{1,n}$ and let $v_{1}v_{2} = e_{12}, v_{3}v_{4} = e_{34}, \ldots, v_{2n-1}v_{2n} = e_{2n-1,2n}$ be the edges of $S_{1,n}$, where $v_{1}$ is the central vertex of $S_{1,n}$. Edges of $BG_3(S_{1,n})$ can be decomposed into subdivision graph of $S_{1,n}$.

**Case i:** $n$ is even

Edges of $BG_3(S_{1,n})$ can be decomposed into $n/2$ paths of length $n/4$ as follows:

$P_{1}: v_1, e_1, e_2, e_1, e_2, \ldots, e_{(n/2)}v_{(n/2)}.$

$P_{2}: v_2, e_1, e_2, e_1, e_2, \ldots, e_{(n/2)}v_{(n/2)}.$

$P_{3}: v_3, e_1, e_2, e_1, e_2, \ldots, e_{(n/2)}v_{(n/2)}.$

$P_{4}: v_4, e_1, e_2, e_1, e_2, \ldots, e_{(n/2)}v_{(n/2)}.$

In two paths $P_{1}$ and $P_{2}$, $v_{1}$ is fixed as a source vertex. If two vertices $v_1$ or $v_1 \in P_{1}$ and $v_1 \in P_{2}$ such that fuzzy shortest path from $e_1$ to $v_1$ or $e_1$ is through $P_{2}$ and fuzzy shortest path from $v_1$ or $e_1$ is through $P_{1}$, then $d (v_1, v_1) = d (v_1, v_1)$ or $d (v_1, v_1) = d (v_1, v_1)$.

Continuing this process for all $n/2$ paths in $BG_3(S_{1,n})$, we get $n/2$ source vertices for $BG_3(S_{1,n})$. Hence, $\tilde{\beta}(BG_3(S_{1,n})) \leq n/2$.

**Case ii:** $n$ is odd

Edges of $BG_3(S_{1,n})$ can be decomposed into $n/2$ fuzzy paths of length four and one fuzzy path of length two as follows:

$P_{1}: v_1, e_1, e_2, e_1, e_2, \ldots, e_{(n/2)}v_{(n/2)}.$

$P_{2}: v_2, e_1, e_2, e_1, e_2, \ldots, e_{(n/2)}v_{(n/2)}.$

$P_{3}: v_3, e_1, e_2, e_1, e_2, \ldots, e_{(n/2)}v_{(n/2)}.$

$P_{4}: v_4, e_1, e_2, e_1, e_2, \ldots, e_{(n/2)}v_{(n/2)}.$

In two paths $P_{1}$ and $P_{2}$, take $e_1$ as a source vertex. If two vertices $v_1$ or $e_1 \in P_{1}$ and $v_1 \in v_1 \in P_{2}$ such that fuzzy shortest path from $e_1$ to $v_1$ or $e_1$ is through $P_{2}$ and fuzzy shortest path from $e_1$ to $v_1$ or $e_1$ is through $P_{1}$, then $d (v_1, v_1)$.
\[ \bar{d} (e_1, e_2) \] or \[ \bar{d} (e_1, e_3) = \bar{d} (e_2, v_i) \] if and only if \( \text{N}(e_1, v_k) = \text{N}(e_2, v_k) \) or \( \text{N}(e_1, e_2) = \text{N}(e_1, v_i) \). This implies that, \[ \beta (BG_3(S_{1,n})) \neq 1. \]

Include \( e_1 \) as another source vertex so that \( \text{N}(e_2, \bar{N}) \neq \text{N}(e_1, e_2) \) or \( \text{N}(e_1, e_3) \neq \text{N}(e_1, v_i) \). \[ \bar{d} (e_1, v_k) \neq \bar{d} (e_2, e_3) \] or \( \bar{d} (e_1, e_3) \neq \bar{d} (e_2, v_i) \).

Continuing this process for all \( n \) paths in \( BG_3(S_{1,n}) \), we get \( n \) source vertices for \( BG_3(S_{1,n}) \). \[ M = \{ e_1, e_3, e_5, \ldots, e_{n-2}, e_n \}. \]

Hence, \[ \beta (BG_3(S_{1,n})) \leq (n+1)/2. \]

IV CONCLUSION

We have determined the fuzzy metric dimension of fuzzy Hypercube \( Q_d \) and \( Q_n \), obtained some new bounds for fuzzy metric dimension of fuzzy Hypercube \( Q_n \).

We have calculated fuzzy metric dimension of fuzzy Boolean graph \( BG_3(G) \) of fuzzy path, fuzzy cycle, star fuzzy graph and \( nK_2 \). We have also determined the fuzzy metric dimension of fuzzy Boolean graph \( BG_3(G) \) of fuzzy path, fuzzy cycle, star fuzzy graph and \( nK_2 \).

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