Flexural Motions of Beams on Foundation Subjected to Moving Concentrated Force

Ogunyebi S.N., Adedowole A., Ogunlade T.O., Oyedele A. A.

Abstract: In this present paper, the dynamic analysis of non-prismatic beams subjected to moving concentrated forces is investigated at constant speed. Two cases of load-beam interaction problems described by the Dirac delta function with constant and harmonic magnitude mobile forces are studied. The technique called Galerkin’s method in conjunction with integral transform method was employed to solve the motion equation. From the numerical results, it is evidently seen that an increase in the foundation stiffness provides reduction on the beam deflection. And furthermore, the issue of resonance is closely monitored and observed to have reached earlier in constant magnitude than harmonic variable magnitude problem. Results presented in this work are useful in constructions engineering designs.

Keywords: Vibrating, Non-Prismatic Beam, Concentrated forces, Harmonic load, foundation, Galerkin method.

I. INTRODUCTION

In the past years, problem of the vibrating motions of elastic structures (such as beam, plates, or shells) subjected to mobile forces has been given considerable attention by several researchers in Physics, Applied Mathematics and related fields. In particular, problem of uniform beams under the passage of moving loads has been studied by numerous authors [1, 2, 3]. Dynamic motions of beam with non-uniformity when subjected to moving load was first addressed by Kolousek et al [4]. Their problems were solved using normal mode analysis. Sadiku and Leipholz [5] examined the dynamics of a prismatic beam and the effect of inertia was considered. Knowles [6] also worked on the dynamic deflection of a beam to a randomly moving load. The analysis of the expected beam deflection and bending moment was carried out when the load was moving with a variable velocity. The inertia effect of the load on the vibration of the system was not considered. Gutierrez and Laura [7] analysed the dynamic of a beam with a non-uniform cross-section and the approximate determination of the dynamic vibration of a beam on a time varying concentrated load. Masoud [8] examined vibration interaction analysis of non-prismatic cross-sectional beam structure under moving vehicle. The problem of a non-uniform cross-section beam with different boundary condition subjected to moving loads, such as a moving concentrated mass and a simple quarter-car (SQC) planar model was studied.

These investigations though impressive, have neglected the more practical situation where the elastic structure are given as variable cross-section. The dynamic response of non-prismatic beam-type resting on variable elastic subgrade and traversed by moving concentrated masses is examined by Ogunyebi et al [9]. In the paper, the flexural effect of rotatory inertia is neglected. This paper presents the problem of vibration analysis of non-prismatic beam with exponential decaying foundation to mobile concentrated masses. The Galerkin’s technique is used in the first instance to lower the order and simplify the motion equation from the fourth order to second order differential equations called Galerkin equations. Analytical solution will be of interest as it sheds light on vital information about the vibrating system.

II. PROBLEM FORMULATION

This section seeks the closed form solutions to the dynamical problem. The equation that governs the system subjected to moving load is given as Frybal [9]

\[ E^* \frac{\partial^4 y}{\partial \xi^4} + \mu(x) \frac{\partial^4 y}{\partial \xi^2 \partial t^2} + K(x) y = P(x,t) \]

\[ + r_0 \mu(x) \frac{\partial^4 y}{\partial \xi^2 \partial t^2} + N \frac{\partial^4 y}{\partial \xi^2 \partial t^2} \]

\[ P(x,t) + r_0 \mu(x,t) \frac{\partial^4 u(x,t)}{\partial \xi^2 \partial t^2} = E_0 \frac{\partial^4 u(x,t)}{\partial \xi^4} + \mu(x) \frac{\partial^4 u(x,t)}{\partial \xi^4} \]

\[ + \mu(x) \frac{\partial^4 u(x,t)}{\partial \xi^4} + E_0 \frac{\partial^4 u(x,t)}{\partial \xi^4} + K(x) u(x,t) \]

(2.0)

where, \( y = u(x,t) \), \( E^* \) is Young’s modulus, \( \mu(x) \) the mass, \( I(x) \) the inertia of the beam, \( K(x) \) is the variable elastic foundation and \( r_0 \) is the measure of rotating inertia. Defining moment of inertia (variable type) and the mass of the beam to be

For the variable moment of inertia \( I \) and the mass per unit length \( \mu \) of the beam adopt the example in [8] and take

\[ I(x) = I_0 (1 + D_0 \sin \frac{\pi x}{L})^3 \]

(2.1)

and

\[ \mu(x) = \mu_0 (1 + D_0) \]

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where

$$\mu(x) = \mu_0(1 + \sin \pi x/L) \quad (2.2)$$

The rigidity of exponential form is given by

$$K(x) = K_0 e^{-\lambda x} \quad (2.3)$$

where \( \lambda \) is a constant and \( K_0 \) is the elastic foundation constant.

### III. BEAM ON EXPONENTIAL DECAY FOUNDATION UNDER CONSTANT MAGNITUDE

In this section, we define

$$P(x, t) = P_o \delta(x - c_m t) \quad (3.0)$$

where \( c_m \) is the velocity of the \( m \)th particles of the system. Equation (3.0) describe the mobile concentrated load and I is taken to be of constant magnitude

Putting equations (2.2), (2.3), (2.4) and (3.0) are substituted into equation (2.0), one obtains

$$EI_o \frac{\partial^2}{\partial x^2} \left[ \left( 1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 u(x,t)}{\partial x^2} \right] + \mu_o \left( 1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$= \mu_o \left( 1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$+ \epsilon_o \left( 1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$+ \epsilon_o \left( 1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\delta u(x,t) - N = 0 \quad (3.1)$$

### IV. METHOD OF SOLUTION

The versatile solution technique called Galerkin’s method to solve the beam problem above is employed. To this end, it is given in the form

$$\begin{align*}
\Box(u) - P_o = 0 \quad \partial u(x,t) - P_o(x,t) = 0
\end{align*} \quad (3.2)$$

Where \( \Box \) represent differential operator, \( u \) the structural deflection and \( P_o \), the load on the beam. The solution of the beam problem is expressed as

$$y = \sum_{i=1}^{n} P_i Q_i \quad (3.3)$$

where \( P_i \) and \( Q_i \) are functions of \( t \) and \( x \) respectively. \( P_i \) is given as modal coordinates and \( Q_i \) is

$$Q_i(x) = \sin \alpha_i + A_i \cos \alpha_i + B_i \sin \alpha_i + C_i \cosh \alpha_i$$

where the constant \( A_i, B_i, \) and \( C_i \) is amplitude of vibration and \( \alpha_i = \pi x / L \).

Noting that the problem at hand is simply supported, equation (3.3) upon consideration of equation (3.4) can be re-written in the form

$$y = \sum_{i=1}^{n} P_i \frac{i \pi x}{L}.$$
By the imposition of orthogonality conditions, equation (3.7) becomes

\[
\sum_{0}^{n} \left[ \frac{E_{l}}{4} \int_{0}^{l} \left( \tilde{R}_{1}(x) \left( \frac{i\pi x}{L} \right)^{2} \sin \frac{i\pi x}{L} - \tilde{R}_{2}(x) \left( \frac{i\pi x}{L} \right)^{2} \sin \frac{i\pi x}{L} \right) \frac{\sin \frac{\pi x}{L}}{L} \right] P_{l}(t) + \mu_{l} R_{l}(x) P_{l}(t) \sin \frac{i\pi x}{L} + \left( \frac{i\pi}{L} \right)^{2} r_{l} \mu R_{1}(x) P_{l}(t) \sin \frac{i\pi x}{L} + N \left( \frac{i\pi}{L} \right)^{2} P_{l}(t) \sin \frac{i\pi x}{L} + \varepsilon_{l}^{\prime} P_{l}(t) \sin \frac{i\pi x}{L} + K_{0} e^{-kx} P_{l}(t) \sin \frac{i\pi x}{L} \right] \left( \frac{k\pi x}{L} \right) \sin \frac{k\pi x}{L} = \int_{0}^{l} P_{l}(x-c_{l}t) \sin \frac{k\pi x}{L} \right] (3.10)
\]

Equation (3.10) after some rearrangements and simplifications yields

\[
D_{1}(i,k) \tilde{H}(t) + D_{2}(i,k) \tilde{E}(t) + D_{3}(i,k) P_{l}(t) = P_{l} \sin \frac{k\pi x}{L} (3.11)
\]

where

\[
D_{1}(i,k) = \mu_{l} \left[ I_{1} + I_{3} \right] + \left( \frac{i\pi}{L} \right)^{2} r_{l} \tag{3.12a}
\]

\[
D_{2}(i,k) = \varepsilon_{l} I_{1} D_{2} = \varepsilon_{l} I_{1} \tag{3.12b}
\]

\[
D_{3}(i,k) = \frac{E_{l} I_{0}}{4} \left[ E_{1} - E_{2} - E_{3} \right] + E_{4} + \frac{E_{l} I_{0}}{4} \left[ E_{1} - E_{2} - E_{3} \right] + E_{4} \tag{3.12c}
\]

\[
E_{1} = \left( \frac{i\pi}{L} \right)^{4} \left[ 10I_{1} - 6I_{2} + 15I_{3} - I_{4} \right] \tag{3.13}
\]

\[
E_{2} = 3i^{3} \left( \frac{\pi}{L} \right)^{4} \left[ 4I_{5} + 5I_{6} - I_{7} \right] \tag{3.14}
\]

\[
E_{3} = 3i^{3} \left( \frac{\pi}{L} \right)^{4} \left[ 8I_{2} - 5I_{3} + 3I_{4} \right] \tag{3.15}
\]

The integrals \( I_{l} \) are as follow

\[
I_{1} = \int_{0}^{l} \sin \frac{\pi x}{L} \sin \frac{k\pi x}{L} \, dx \tag{3.16a}
\]

\[
I_{2} = \int_{0}^{l} \cos \frac{\pi x}{L} \sin \frac{\pi x}{L} \, dx \tag{3.16b}
\]

\[
I_{3} = \int_{0}^{l} \sin \frac{\pi x}{L} \sin \frac{k\pi x}{L} \, dx \tag{3.16c}
\]

\[
I_{4} = \int_{0}^{l} \cos \frac{\pi x}{L} \sin \frac{k\pi x}{L} \, dx \tag{3.16d}
\]

\[
I_{5} = \int_{0}^{l} \sin \frac{\pi x}{L} \sin \frac{k\pi x}{L} \, dx \tag{3.16e}
\]

\[
I_{6} = \int_{0}^{l} \cos \frac{\pi x}{L} \sin \frac{k\pi x}{L} \, dx \tag{3.16f}
\]

\[
I_{7} = \int_{0}^{l} \frac{3\pi x}{L} \cos \frac{\pi x}{L} \sin \frac{k\pi x}{L} \, dx \tag{3.16g}
\]

Evaluating the integrals in (3.16a), equations (3.12a) to (3.15) yield the following respectively

\[
E_{1} = i^{3} \left( \frac{\pi}{L} \right)^{4} \left[ \frac{60iL}{2} \right] \tag{3.17}
\]

\[
E_{2} = 3i^{3} \left( \frac{\pi}{L} \right)^{4} \left[ \frac{60iL}{2} \right] \tag{3.18}
\]

\[
E_{3} = 3i^{3} \left( \frac{\pi}{L} \right)^{4} \left[ \frac{60iL}{2} \right] \tag{3.19}
\]

And
At this stage, equation (3.11) is subjected to a Laplace transform and after further simplification give

\[(3.21b)\]

which reduces to

\[(3.23)\]

where

\[
\beta_1 = \frac{-D_2 + \sqrt{D_2^2 - 4D_1D_3}}{2D_1}
\]

\[
\beta_2 = \frac{-D_2 - \sqrt{D_2^2 - 4D_1D_3}}{2D_1}
\]

Adopting the representation below, the Laplace inversion of equation (3.24) is given as

\[g(s) = \frac{\theta}{S^2 + \theta^2}, f_1(s) = \frac{1}{S - \beta_1}\]

and

\[f_2(s) = \frac{1}{S - \beta_2}\]

Further simplification and arrangements of (3.26) gives

\[P_i(t) = P_p \left[ \frac{e^{\beta_1 t}}{\beta_1} - \frac{e^{\beta_2 t}}{\beta_2} - \frac{1}{\beta_2} \right]
\]

where

\[P_p = \frac{P_0}{(\beta_1 + \beta_2)} \quad P_p = \frac{P_0}{\beta_1 \beta_2}
\]

\[I_1 = \int_0^t e^{-\beta_1 u} \sin \theta du \quad \text{and} \quad I_2 = \int_0^t e^{-\beta_2 u} \sin \theta du
\]

Evaluating the integrals (3.29) above, we obtain

\[I_1 = \frac{1}{(\theta^2 + \beta_1^2)} (\theta e^{-\beta_1 t} \cos \theta t + \theta_1 e^{-\beta_1 t} \sin \theta t)
\]

\[I_2 = \frac{1}{(\theta^2 + \beta_2^2)} (\theta e^{-\beta_2 t} \cos \theta t + \beta_1 e^{-\beta_2 t} \sin \theta t)
\]

Subjecting equation (3.27) to some simplification and rearrangement yields,

\[P_i(t) = \frac{P_p}{\beta_1 (\theta^2 + \beta_1^2)} \left( \theta (e^{\beta_1 t} - \cos \theta t) - \beta_1 e^{-\beta_1 t} \sin \theta t \right)
\]

which on inversion yields

\[u(x,t) = \sum_{i=1}^n \left\{ \frac{P_p}{\beta_1 (\theta^2 + \beta_1^2)} \left( \theta (e^{\beta_1 t} - \cos \theta t) - \beta_1 e^{-\beta_1 t} \sin \theta t \right) \right\} \sin \frac{\pi x}{L}
\]

which is the solution of the governing differential equation of the non-prismatic element at constant mobile distributed masses on exponential decaying subgrade.
V. BEAM ON EXPONENTIAL DECAY FOUNDATION UNDER HARMONIC MAGNITUDE

This section seeks the solution of the non-prismatic structure with exponentially decaying foundation under harmonic magnitude at constant speed. The load on the structure is given as

\[ P(x,t) = P_o \cos \omega t \delta (x - c_n t) \quad (4.1) \]

Using equation (2.0) in (4.1), the differential equation gives

\[ EI(x) \frac{d^4 u_n(x,t)}{dx^4} + \mu(x) \frac{d^4 u_n(x,t)}{dx^4} - \rho c_n \frac{d^4 u_n(x,t)}{dx^4 dt^2} + \varepsilon_n \frac{\partial u_n}{\partial t} + K(x) u_n(x,t) = P_o \cos \omega t \delta (x - c_n t) \quad (4.2) \]

\[ u_n(x,t) = \sum_{m=1}^{\infty} P_m(t)Q_m(x) \quad (4.3) \]

And for the simply supported study, equation (4.3) is rewritten in the form

\[ u_n(x,t) = \sum_{m=1}^{\infty} P_m(t) \sin \frac{\pi n x}{L} \quad (4.4) \]

Putting equation (4.4) into equation (4.2) and using the same procedures as in section (3.0) one obtains

\[ \sum_{n=1}^{\infty} \left[ D_1 \frac{d^2 P_m}{dt^2} + D_2 \frac{dP_m}{dt} + D_3 P_m \right] = P_o \cos \omega t \sin \theta \quad (4.5) \]

where \( c_n \) is the speed and \( P_n(t) \).

Considering the mth particle of the system under consideration, equation (4.5) becomes

\[ D_1 \frac{d^2 P_m}{dt^2} + D_2 \frac{dP_m}{dt} + D_3 P_m = P_o \cos \omega t \sin \theta \quad (4.6) \]

Subjecting equation (4.6) as defined previously yields

\[ y_n(S) = \frac{P_o}{2(d_1 - d_2)} \left( \frac{\Omega_1 \alpha_1}{S^2 + \Omega_1^2} - \frac{\alpha_1}{S - \alpha_1} - \frac{\Omega_2 \alpha_2}{S^2 + \Omega_2^2} - \frac{\Omega_1 \alpha_1}{S^2 + \Omega_1^2} + \frac{\alpha_1}{S - \alpha_1} - \frac{\Omega_2 \alpha_2}{S^2 + \Omega_2^2} - \frac{\alpha_2}{S - \alpha_2} \right) \quad (4.7) \]

where

\[ \Omega_1 = \omega + \frac{k \pi c_m}{L} \quad \Omega_2 = \omega - \frac{k \pi c_m}{L} \quad (4.8) \]

\[ \alpha_1 = D_2 + \frac{\sqrt{D_2^2 - 4D_1 D_3}}{2D_1} \quad (4.9a) \]

\[ \alpha_2 = D_2 - \frac{\sqrt{D_2^2 - 4D_1 D_3}}{2D_1} \quad (4.9b) \]

Following the argument in section (3.0), equation (4.6) gives

\[ P_n(t) = \frac{P_o}{\beta_1 (\Omega_1^2 + \beta_1^2)} \left( \Omega_1 \left( e^{\beta_1 t} - \cos \Omega_1 t \right) - \beta_1 \sin \Omega_1 t \right) \]

\[ - \frac{P_o}{\beta_2 (\Omega_2^2 + \beta_2^2)} \left( \Omega_2 \left( e^{\beta_2 t} - \cos \Omega_2 t \right) - \beta_2 \sin \Omega_2 t \right) \]

\[ - \frac{P_o}{\beta_3 (\Omega_3^2 + \beta_3^2)} \left( \Omega_3 \left( e^{\beta_3 t} - \cos \Omega_3 t \right) - \beta_3 \sin \Omega_3 t \right) \]

\[ + \frac{P_o}{\alpha_1 (\alpha_1^2 + \beta_1^2)} \left( \Omega_1 \left( e^{\alpha_1 t} - \cos \Omega_1 t \right) - \alpha_1 \sin \Omega_1 t \right) \]

\[ + \frac{P_o}{\alpha_2 (\alpha_2^2 + \beta_2^2)} \left( \Omega_2 \left( e^{\alpha_2 t} - \cos \Omega_2 t \right) - \alpha_2 \sin \Omega_2 t \right) \]

\[ + \frac{P_o}{\alpha_3 (\alpha_3^2 + \beta_3^2)} \left( \Omega_3 \left( e^{\alpha_3 t} - \cos \Omega_3 t \right) - \alpha_3 \sin \Omega_3 t \right) \quad (4.10) \]

which on inversion yields

\[ u_n(x,t) = \sum_{m=1}^{\infty} \left( \frac{P_o}{\alpha_1 (\alpha_1^2 + \beta_1^2)} \left( \Omega_1 \left( e^{\alpha_1 t} - \cos \Omega_1 t \right) - \alpha_1 \sin \Omega_1 t \right) \right) \sin \frac{\pi n x}{L} \quad (4.11) \]

which is the deflection response of the structure subjected to actions of harmonic magnitude load at constant velocity.

VI. DISCUSSION

Effects of Resonance

The dynamic effect of resonance conditions is paramount in the study of dynamic system such as beam problem. These are the conditions under which the non-prismatic beam response grow without bound. In studying a dynamic problem such as this, it is desirable to examine the phenomenon of resonance condition. These are the conditions under which the non-prismatic beam responses grow without bound.

Evidently, equation (3.34) shows that the non-prismatic beam response under constant magnitude mobile concentrated forces will grow without bound whenever

\[ \beta_1 = \beta_2, \quad \theta^2 = -\beta_1^2, \quad \text{or} \quad \theta^2 = -\beta_2^2 \quad (5.1) \]

And the velocity at which this occurs is termed the critical condition, while from equation (4.8) the same non-prismatic beam traversed by harmonic variable magnitude mobile forces will experience resonance effects whenever

\[ \alpha_1 = \alpha_2, \quad \Omega_1^2 = -\alpha_1^2, \text{or} \quad \Omega_2^2 = -\alpha_2^2 \quad (5.2) \]

It can be deduced by equations (5.1) and (5.2) that the critical velocity of the structure when on exponentially decaying subgrade is smaller than when the system involving non-pragmatic beam subjected to harmonic variable magnitude mobile force. Therefore, resonance is reached earlier in constant magnetic than harmonic variable magnitude.
VII. NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

As an illustration, the non-prismatic beam is taken to be 12.192m long and the mass is taken to travel at the constant velocity 8.12m/s. Furthermore, we assumed that force travels at the EI is 6068242m²/s². The deflection of the beam with exponential decaying foundation for the values of K and R₀ are plotted against t. The analysis was carried out separately for both cases of constant magnitude and harmonic magnitude of mobile forces problems.

Figure 1 displays the effect of foundation modulus $K₀K₀$ on the transverse deflection of non-prismatic elastic beam when traversing concentrated forces are of constant magnitudes. The figure depict that as $K₀$ increases the displacement response of beam decreases.

Figure 2 depicts the deflection of FM for constant force for fixed value of Rotatory inertia $R₀$ and axial force $N$. It is seen that an increase in FM reduces the deflection profile of the beam.

Figure 3 shows deflection of axial force on the transverse deflection of non-prismatic elastic beam when traversing concentrated forces is of constant magnitudes. An increase in $N$ decreases the deflection of the elastic beam.

Figure 4 displays the response amplitude of the harmonic mobile forces. It is clearly seen the response amplitude decreases as rotatory inertia increases.

Figure 5 displays the deflection of the harmonic mobile forces for the beam. The response amplitude decreases as axial force $N$ increases.

Figure 7 compares the deflection profiles of the non-prismatic structure for constant magnitude and harmonic magnitude at constant speed for fixed values of FM=4000000, N=500000 K₀ 0.5. Evidently, the deflection amplitude of constant magnitude mobile load is higher than that of the harmonic variable mobile forces.
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