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Discrete- Time Queueing Model $Geo^X/G/\infty$ with Bulk Arrival Rule

Pukazhenthi. N, Ramki. S

Abstract: ***A discrete time queueing model $Geo^X/G/\infty$ is considered to estimate of the number of customers in the system. The arrivals, which are in groups of size X, inter-arrivals times and service times are distributed independent. The inter-arrivals fallows geometric distribution with parameter p and service times follows general distribution with parameter μ, we have derive the various transient state solution along with their moments and numerical illustrations in this paper.***

Key word: Discrete time, Customers, Group size, Inter-arrival, Transient distributions, Geometric distribution.

I. INDRODUCTION

In this paper, we consider a discrete time queueing model $Geo^X/G/\infty$, to find the solution of the numbers of customers in the system. The customer that are stay in the system or depart from the system at time k, in which arrivals are in groups of size X. The inter-arrival time distributed independently with geometrical distribution and service time distributed independently with general distribution. In addition, to that it is difficult to solve the system $Geo^X/G/c$, such a system can be approximated by $Geo^X/G/\infty$if we take c as large and ρ as small. Let the time – axis is split-up into interval as $0,\delta,2\delta,\dots,m\delta,\dots$,. For the sake of simplicity, we assume $\delta = 1$. Consider in each epoch n the arrivals occur in $n-,n$ and departures in $n,n+$.

In the past decades many service systems, where the number of servers facilities arranged in parallel is large like infinite server systems. Takagi, H.[4] provide an queueing analysis –. a foundation of performance evaluation: discrete time systems. Neuts, M.[1] provide an Matrix-Geometric Solutions in Stochastic Models. Sivasamy, R and Elangovan, R. [8] discuss a bulk service queues of M/G/1 type with accessible batches and single vacation. Tijms, H.C [3] has discussed Stochastic Modelling and Analysis: A computational Approach. Jain,G., and Sigman, K [5] are studied a Pollaczek-Khinitchine formula for M/G/1 Queues with disasters. Latouche, G., and Ramaswami, V [7] have presented introduction to matrix analytic methods in Stochastic Modelling. Goswami, V. and Gupta,U.C [6] have discussed the discrete-time Multiserver queue Geom/Geom/m Queue with Late and Early Arrivals. Yi, X.W., Kim, J.D *et al.*,[9] are studied the Geo/G/1 Queue with disasters and multiple working vacations. Holman, D.F *et al.*, [2] are providing on the service system MA/G/cc. Sivasamy,R and Pukazhenthi,N.[10] have discussed a discrete time bulk service queue with accessible batch: Geo/NB(L, K) /1. N. Pukazhenthi and S.Ramki [12] have denoted the performance analysis of discrete time bulk service queuing model NB (L,K)/Geo/1. N.Pukazhenthi and M.Ezhilvanan [11] are carried out the analysis of discrete time queues with single server using correlated times. N.Pukazhenthi and S.Ramki [13] are studied analysis of discrete time NB/Geo/1 queuing model with system capacity (L,K).

The potential applications of the model in this study are a self-service system, in which a customer is always accompanied by a server that is himself, modern technological service systems such as stock exchange where people are making demands to get information, etc. From the above systems, the customers can usually occur in groups rather than single. Now we state the assumptions of the model and then their moments and steady state solutions.

II. MODEL DESCRIPTION

The discrete-time queueing model $\mathrm{Geo^X/G/\infty}$ follows a geometric distribution with the parameter p and mean $\frac{1}{p}$ and the probability function (pmf).

$$P(X = m) = a_m, \qquad m = 1,2,\dots\infty$$

Let $\mathcal{A}\{z\}$ be the size of group of pgf by which

$$\mathcal{A}\{z\} = E(Z^X) = \sum_{m=1}^{\infty} a_m z^m \qquad \dots(1)$$

which $\mathcal{A}^1\{1\} = \sum_{m=1}^{\infty} m a_m$

$$\frac{\mathcal{A}^{(k)}\{z\}}{k!} = \sum_{m=k}^{\infty}\binom{m}{k} a_m z^{m-k}$$

The kth derivative of $\mathcal{A}\{y\}$ is identified by $\mathcal{A}^k\{z\}$

If the size of the ith group be X_i then $\mathcal{A}\{k\}$ be the number of customer arrives during (0,k) is derived as

$$\mathcal{A}\{k\} = \sum_{i=1}^{k} X_i$$

On arrival of the customers they start the service immediately and according to the general distribution of the service time s of an individual customer are independently identical distributed with mean $\frac{1}{\mu} = \sum_{i=1}^{\infty} i b_i$.

$$B(j) = P(S \le j) = \sum_{i=1}^{j} b_i \qquad \dots(2)$$

Let us consider at time k = 0 there are $I \ge 0$ customers available in the service are,

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* Correspondence Author
Pukazhenthi. N*, Assistant professor, Department of statistics, Annamalai University, chidambarm, Tamilnadu pukazhstat@gmail.com
Ramki. S, Research scholar, Department of statistics, Annamalai University, Chidambaram, Tamilnadu, ramki.stat24@gmail.com

$$P(I = i) = d_i, \quad \sum_{i=0}^{\infty} d_i = 1, \bar{d} = \sum_{i=1}^{\infty} id_i$$

The random variables are defined as follows.

$N(k) =$ Total number of customers in system at time k.

$\mathcal{A}(k) =$ Total number of customer that arrive during (0,k].

$C(k) =$ Total number of customers that depart during (0,k] out of the customers that arrive during the same period

$N_1(k) =$ Total number of customers in system at time k out of the initial number I in the system at time zero

$X^n(k) =$ Total number of customer that stay in system at time k out of n groups that arrive during (0,k].

$N_2(k) =$ Total number of customers that stay in system at time k out of the total number that arrive during (0,k].

$$= \sum_{n=0}^{k} X^n(k), \qquad k = 1,2,\dots$$

Clearly,

$$\mathcal{A}(k) = N_2(k) + C(k)$$

and

$$Z(k) = N_1(k) + N_2(k)$$

III. DERIVATIONS OF VARIOUS DISTRIBUTIONS

Distribution of number of busy servers.

First, we consider the distribution of $N_1(k)$. If out of the initial number I at time $k = 0, n$ are left in the system by time k, then

$$P_{N_{1}(k)}(n) = (PN_1(k) = n) = \sum_{i=0}^{\infty} \binom{i}{n} (B(k))^{i-n} (1 - B(k))^n d_i, \quad 0 \le n \le i \qquad \dots(3)$$

It is easy to see that pgf of $N_1(k)$ is

$$P_{N_{1}(k)}(n) = \sum_{n=0}^{i} P_{N_{1}(k)}(n)\, z^n$$
$$= \sum_{i=0}^{\infty} [\,(1 - B(k))z + B(k)]^i d_i \qquad \dots(4)$$

which gives mean $EN_1(k) = \bar{d}[1 - B(k)]$

and variance

$$Var\big(N_1(k)\big) = \sum_{i=1}^{\infty} \{i^2(1 - B(k))^2\}\, d_i + B(k)E\big(N_1(k)\big) - \Big(E\big(N_1(k)\big)\Big)^2$$

If $k \to \infty$, both mean and variance tend to 0, as expected. In fact, $\lim_{k\to\infty} P_{N_{1}(k)}(0) = 1$. Out of the initial number I at time zero, the distribution of the number that departs during $0, k]$ can be discussed similarly.

Next, we study the distributions of $X^{(1)}(k)$ and $\mathcal{A}(k)$.

First, note that, $X^r(k) =$ r independently identically distributed random variables each distributed as $X^{(1)}(k)$.

Then, using the fact that the arrival time of a group is uniformly distributed over k intervals,

$p_n(k) = P(X^{(1)}(k) = n)$

$= \sum_{m=1}^{k} P(S > k - m$ for n customers from a group arriving at epoch m) … (5)

$= \sum_{m=1}^{k} \sum_{l=1}^{\infty} P(S > k - m$ for n customers of a group of size $l|$, a group of size arrives $l$at epoch $m)a_l$

$$= \frac{1}{k} \sum_{l=1}^{\infty} \sum_{m=1}^{k} \binom{l}{n} (1 - B(k - r))^n (B(k - r))^{l-n} a_l \qquad \dots(6)$$

$$= \frac{1}{k} \sum_{l=1}^{\infty} \sum_{j=0}^{k-1} \binom{l}{n} (1 - B(j))^n (B(j))^{l-n} a_l, \quad 1 \le n \le l.$$

The pgf of $p_n(k)$ is

$$\bar{P}_k(z) = \sum_{n=0}^{l} p_n(k) z^n$$
$$= \frac{1}{k} \sum_{j=0}^{k-1} \mathcal{A}\{(1 - B(j))z + B(j)\} \qquad \dots(7)$$

Also, Since

$$k_j^{(k)} = P\left(\sum_{i=1}^{k} X_i = j\right),$$

Represents the distribution of the number of customers that arrive during $(0.k)$, the pgf of $k_j^{(k)}$, using the concept of convolution, is given by

$$\bar{K}^{(k)}(y) = \sum_{j=0}^{\infty} k_j^{(k)} z^j$$

$$= (1 - p + p\mathcal{A}\{z\})^k \qquad \dots(8)$$

$$= \sum_{m=0}^{k} \binom{k}{m} p^m (1 - p)^{k-m} (\mathcal{A}\{z\})^m \qquad \dots(9)$$

Where $1 - p + p\mathcal{A}(z)$ is the pgf of one of the independent and identically distributed random variable's $X_1, X_2, \dots, X_k$. Note that $k_j^{(k)}$ is the coefficient of z^j in (9). Also, note that $P\big(k_j^{(0)} = 0\big) = 1$, Since, by assumption, the new arrivals are permitted after $k = 0$.

Probability generating function of $N_2(k)$.

If $s\ (0 \le s \le k)$ groups arrive in $(0, k]$, then the pgf of $N_2(k)$is

$$\bar{P}_{N_{2}(k)}(Z) = \sum_{s=0}^{k} \left(\frac{1}{k} \sum_{j=0}^{k-1} \mathcal{A}\{1 - B(j))z + B(j)\}\right)^s \binom{k}{s} p^s (1 - p)^{k-s} \qquad \dots(10)$$

$$= \left[\frac{p}{k} \sum_{j=0}^{k-1} \mathcal{A}\{(1 - B(j))z + B(j)\} + 1 - p\right]^k$$

Were we have used (7).

Finally, since

$$N(k) = N_1(k) + N_2(k)$$

The use of convolutions gives

$$P_{N_k}(n) = P(N(k) = n)$$
$$= \sum_{r=0}^{n} P_{N_{1}(k)}(m) P_{N_{2}(k)}(n - m) \qquad \dots(11)$$

Also, the transform of the distribution C(k) is given by.

$$\bar{P}_{C_k}(z) = \left[\frac{p}{k} \sum_{j=0}^{k-1} \mathcal{A}\{B(j)z + \big(1 - B(j)\big)\} + 1 - p\right]^k \qquad \dots(12)$$

IV. MOMENTS

The performance measures for various distributions can be obtained as follows:

(i) The expected value of $N_2(k)$ is given by

$$E(N_2(k)) = \overline{P}^{(1)}_{N_2\,(k)}$$

$$= p\bar{a}\sum_{j=0}^{k-1}\sum_{i=j+1}^{\infty} b_i \quad \ldots (13)$$

To get the variance of $N_2(k)$, we first get $\overline{P}^{(2)}_{N_2\,(k)}(1)$ which is given by

$$\overline{P}^{(2)}_{N_2\,(k)}(1) = k(k-1)\left[\frac{p\bar{a}}{k}\sum_{j=0}^{k-1}(1-B(j))\right]^2 + p\left(\sum_{j=0}^{k-1}(1-B(j))^2\right)\mathcal{A}^{(2)}(1)$$

and hence

$$\begin{aligned}Var(N_2(k)) &= \overline{P}^{(2)}_{N_2\,(k)}(1) + \overline{P}^{(1)}_{N_2\,(k)}(1) - \left(\overline{P}^{(1)}_{N_2\,(k)}(1)\right)^2 \\ &= -\frac{1}{k}[E(N_2(k))]^2 + E(N_2(k)) \\ &\quad + p\left(\sum_{j=0}^{k-1}(1-B(j))^2\right)\mathcal{A}^{(2)}(1) \quad \ldots (14)\end{aligned}$$

If we let $k \to \infty$ and take appropriate limits it can be seen that the discrete time results given in (13) and (14) match the continuous – time results.

(ii) The moments of $C(k)$ can be obtained similarly and are given by

$$E(D(k)) = \overline{P}^{(1)}_{C\,(k)}(1) = p\bar{a}\sum_{j=0}^{k-1}\sum_{i=1}^{j} b_i \ldots (15)$$

and

$$\begin{aligned}Var(C(k)) &= \overline{P}^{(2)}_{C\,(k)}(1) + \overline{P}^{(1)}_{C\,(k)}(1) - (\overline{P}^{(1)}_{C\,(k)}(1))^2 = -\tfrac{1}{k}[E(C(k))]^2 + E(C(k)) \\ &\quad + p\left(\sum_{j=0}^{k-1}(B(j))^2\right)\mathcal{A}^{(2)}(1) \quad \ldots (16)\end{aligned}$$

(iii) The mean and variance of $\mathcal{A}(k)$ can be obtained from (8) and are given by

$$E(\mathcal{A}(k)) = p\bar{a}k, \quad \ldots (17)$$

and

$$var(\mathcal{A}(k)) = p\bar{a}k(1-p\bar{a}) + pk\mathcal{A}^{(2)}(1) \quad \ldots (18)$$

respectively, from (13), (15) and (17),

we note that

$$E(\mathcal{A}(k)) = E(N_2(k)) + E(C(k)) \quad \ldots (19)$$

However, one can get $Cov(N_2(k), C(k))$ from

$$\begin{aligned}var(\mathcal{A}(k)) &= Var(N_2(k)) + Var(C(k)) \\ &\quad + 2\,Cov(N_2(k), C(k)) \quad \ldots (20)\end{aligned}$$

5. STEADY – STATE PROBABILITY

From (10), we get

$$\overline{P}_{N_2}(z) = \lim_{k\to\infty}\overline{P}_{N_2\,(k)}(z)$$

$$= \lim_{k\to\infty}\left[\frac{p}{k}\sum_{j=0}^{k-1}\mathcal{A}\{B(j) + (1-B(j))z\} + 1 - p\right]^k$$

$$= \lim_{k\to\infty}\left[1 - \frac{p}{k}\sum_{j=0}^{k-1}[1 - \mathcal{A}\{B(j) + (1-B(j))z\}\right]^k$$

$$= \exp\left(p\sum_{j=0}^{\infty}[\mathcal{A}\{B(j) + (1-B(j))z\} - 1]\right) \ldots (21)$$

For numerical computation, we rewrite (21) as

$$\overline{P}_{N_2}(z) = 1 + \phi(z) + \frac{\phi^2(z)}{2!} + \frac{\phi^3(z)}{3!} + \cdots$$

Where $\phi(z)$ is a polynomial defined as

$$\phi(z) = p\sum_{j=0}^{\infty}[\mathcal{A}\{B(j) + (1-B(j))z\} - 1]$$

6. COMPUTATIONS OF $PN_{(k)}(n)$, AND $PC_{(k)}(n)$

To get $P_{N\,k}(n)$, get $P_{N_1\,(k)}(n)$ from (3), $P_{N_2\,(k)}(n)$ from (10), and finally, $P_{N\,(k)}(n)$ from (11).

Since the difficult part here is to get $P_{N_2\,(k)}(n)$ we outline below an efficient algorithmic procedure for this. Note that $P_{N_2\,(k)}(n)$ = coefficent of z^n in (10)

$$= \sum_{s=0}^{k-1}(h(N))^s\binom{k}{s}p^s(1-p)^{k-s}$$

where

$$h(N) = \frac{1}{k}\sum_{j=0}^{k-1}\mathcal{A}\{1 - B(j)z + B(j)\}$$

$$= \frac{1}{k}\sum_{l=1}^{\infty}\sum_{n=0}^{l}\sum_{j=0}^{k-1}\binom{l}{n}(1-B(j))^n (B(j))^{l-n}a_l z^n \quad \ldots (22)$$

Since (22) gives the coefficient of y^n in $h(z)$, we can find the coefficient of y^n in $(h(z))^s$ as follows:

Let

$$\sum_{n=0}^{\infty} C_{sn}z^n = (h(z))^s, \quad 0 \le s \le k$$

So that

$$h_{00} = 1 \text{ and } h_{0n} = 0, \quad n \ne 0$$

and

$$h_{jn} = \sum_{i=0}^{n} h_i h_{j-1,\; n-i}, \quad j \ge 1, n$$

with

$$h_i = \frac{1}{k}\sum_{l=1}^{i}\sum_{j=0}^{k-1}\binom{l}{i}(1-B(j))^i (B(j))^{l-i}a_l, 0 \le i \le l.$$

For $P_{C\,k}(n)$, we rewrite (12) as

$$P_{C\,k}(n) = \text{coefficent of } z^n \text{ in } (12)$$

$$= \sum_{s=0}^{k-1}(d(z))^s\binom{k}{s}p^s(1-p)^{k-s}$$

where

$$d(z) = \frac{1}{k}\sum_{j=0}^{k-1}\mathcal{A}\{B(j)z + 1 - B(j)\}$$

N	$K = 5$	$K = 10$
0	0.4344	0.2964
1	0.2748	0.2354
2	0.1639	0.1822
3	0.0710	0.1325
4	0.0274	0.0936
5	0.0066	0.0619
6	0.0016	0.0404
7	0.0003	0.0254
8	0.0000	0.0189
9	0.0000	0.0128
10	0.0000	0.0060
Size	11	21
Sum	1.000	1.000
$E\,\mathcal{A}(k)$	1.020	3.570
$\sigma\mathcal{A}(k)$	1.147	2.213

$$= \frac{1}{k}\sum_{l=1}^{\infty}\sum_{n=0}^{l}\sum_{j=0}^{k-1}\binom{l}{n}(B(j))^n(1-B(j))^{l-n}a_l z^n$$

and proceed as in the case of $P_{N_{2\,(k)}}(n)$.

V. NUMERICAL RESULTS

The above algorithms were used to test several cases, but

results are being presented for one particular case. They are shown in self-explanatory tables.

Table-1

Distributions of $P_{\mathcal{A}(k)}\ (n)$ for the model $Geo^X/G/\infty$, n=10, p = 0.3 $b_1 = b_2{=}b_3 = b_4 = b_5 = 0.2,\ \ b = 3.$

N	$K = 5$	$K = 10$
0	0.2981	0.3282
1	0.2308	0.2663
2	0.1799	0.1958
3	0.1332	0.1353
4	0.0965	0.0865
5	0.0605	0.0631
6	0.0329	0.0474
7	0.0120	0.0264
8	0.0076	0.0158
9	0.0009	0.0041
10	0.0004	0.0001
Size	11	21
Sum	1.000	1.000
$E\,\mathcal{A}(k)$	2.550	5.100
$\sigma\mathcal{A}(k)$	1.830	2.588

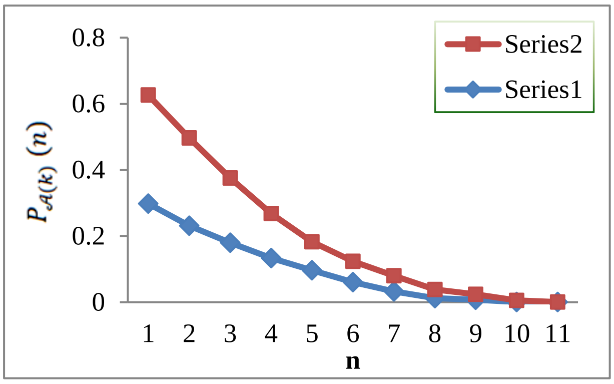

Fig-1

Table-2

Distributions of $P_{C(k)}\ (n)$ for the model $Geo^X/G/\infty$ n = 10, p = 0.3, $b_1 = b_2{=}b_3 = b_4 = b_5 = 0.2, b = 3.$

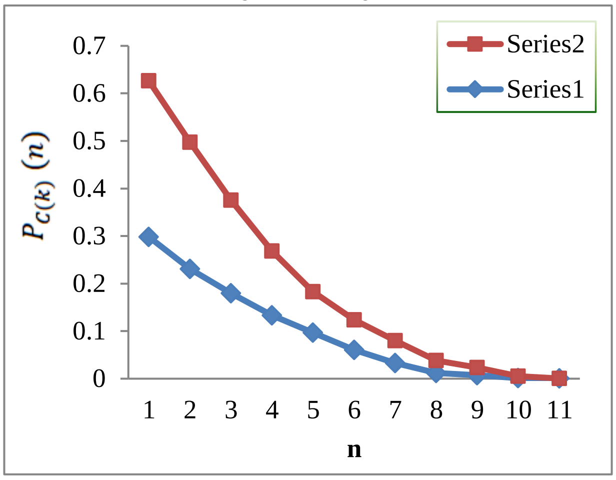

Fig-2

Table-3

Distributions of $P_{N_2(k)}\ (n)$ for the model $Geo^X/G/\infty$, $n = 10$, $p = 0.3$, $b_1 = b_2{=}b_3 = b_4 = b_5 = 0.2, b = 3.$

N	$K = 5$	$K = 10$
0	0.3008	0.3232
1	0.2118	0.2193
2	0.1381	0.1541
3	0.0899	0.1042
4	0.0476	0.0781
5	0.0218	0.0556
6	0.0079	0.0408
7	0.0017	0.0233
8	0.0004	0.0114
9	0.0000	0.0063
10	0.0000	0.0001
Size	11	21
Sum	1.000	1.000
$E\,\mathcal{A}(k)$	1.020	3.570
$\sigma\mathcal{A}(k)$	1.147	2.213

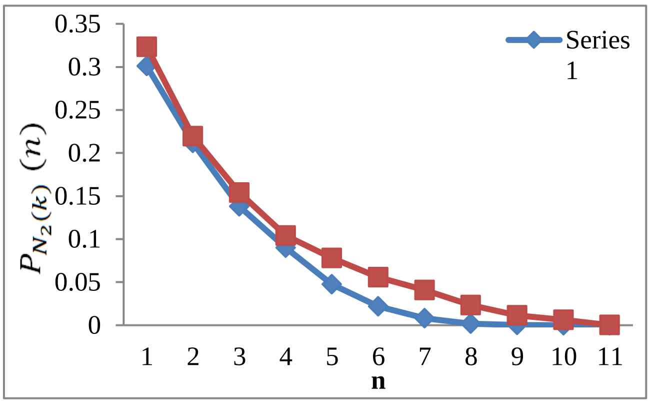

Fig-3

Table-4

Distributions of $P_{N(k)}\ (n)$ for the model
$Geo^X/G/\infty$, n=10, p = 0.3,
$b_1 = b_2 = b_3 = b_4 = b_5 = 0.2$, $b = 3$.

N	*K = 5*	*K = 10*
0	0.3208	0.3432
1	0.2518	0.2293
2	0.1981	0.1341
3	0.1499	0.0842
4	0.1076	0.0381
5	0.0718	0.0156
6	0.0379	0.0108
7	0.0214	0.0053
8	0.0104	0.0011
9	0.0058	0.0003
10	0.0000	0.0001
Size	11	21
Sum	1.000	1.000
$E\ \mathcal{A}(k)$	1.020	3.570
$\sigma\mathcal{A}(k)$	1.147	2.213

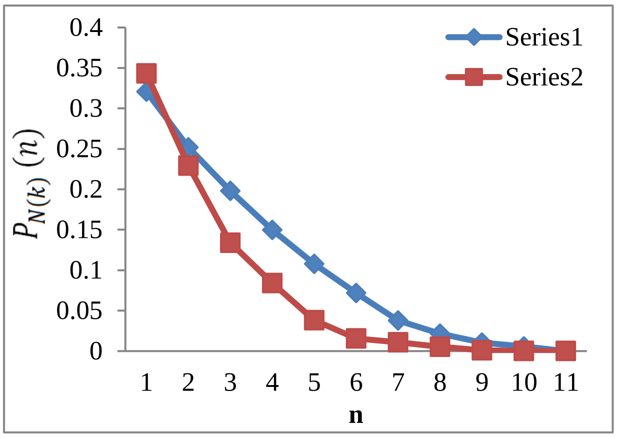

Fig-4

VI. CONCLUSION

At conclusion, for the queueing system $Geo^X/G/\infty$ we have derived various transient distributions along with their moments and also their numerical computation. For more general cases the results can be derived easily by changing "p" to "p_i" for the inputs of time dependent. In which "i" is an arrival epoch. In future the result of this paper can be further extended to include the case of correlated arrival.

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