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Fine Fuzzy 𝔰𝔭 Closed Sets in Fine Fuzzy Topological Spaces

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Abstract: The main view of this article is the extended version of the fine topological space to the novel kind of space say fine fuzzy topological space which is developed by the notion called collection of quasi coincident of fuzzy sets. In this connection, fine fuzzy 𝔰𝔭closed sets are introduced and studied some features on it. Further, the relationship between fine fuzzy 𝔰𝔭closed sets with certain types of fine fuzzy closed sets are investigated and their converses need not be true are elucidated with necessary examples. Fine fuzzy 𝔰𝔭continuous function is defined as the inverse image of fine fuzzy closed set is fine fuzzy 𝔰𝔭closed and its interrelations with other types of fine fuzzy continuous functions are obtained. The reverse implication need not be true is proven with examples. Finally, the applications of fine fuzzy 𝔰𝔭continuous function are explained by using the composition.

Keywords: Fine fuzzy topological space, Fine fuzzy 𝔰𝔭closed set, Fine fuzzy continuous, Fine fuzzy 𝔰𝔭**continuous. Subject Classification Primary: 54A05, 54A10, 54A20.

I. INTRODUCTION

The Fuzzy set theory uses the linguistic variable to represents imprecise concepts in many real-life applications in engineering, robotics, spacial objects, biosciences, etc. It is a marvelous tool for modeling in the various kinds of uncertainty associated with imprecision and vagueness. American Cybernatist Lofti A. Zadeh [12] initiated the theory of fuzzy set in 1965. Later, in the year 1968, the Chang [3] extended the notion of a fuzzy topology. Further, Power P. L. and Rajak K[9] investigated fine topological space, was particular case of generalized topological space. The views on sp-open sets in general topology and fuzzy sp-open sets were introduced by Dontcev, Przemski in 1996 [5] and Hakeem A. Othman [6] in 2011 respectively. This paper is mainly focused to the extension of fine topological space. The approach on fine fuzzy 𝔰𝔭 closed sets in fine fuzzy topological spaces were well developed and studied. Further, the interrelations of fine fuzzy 𝔰𝔭 closed sets with distinctive types of fine fuzzy closed set were investigated. Later, we defined the fine fuzzy 𝔰𝔭 continuous functions and briefly discussed on its properties.

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II. PRELIMINARIES

Definition 2.1[11]

Let X be a space of points and I be the unit interval [0, 1]. A fuzzy set λ in X is a mapping from X → I

Definition 2.2[3]

A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions

(i) 0 and 1 ∈ T.
(ii) If $\mu, \delta \in$T, then $\mu \wedge \delta \in$ T.
(iii) If $\mu_i \in$ T for each i ∈ I then $\vee \mu_i \in$ T, then T is called a fuzzy topology and the pair (X, T) is a fuzzy topological space.

Definition 2.3[3]

Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(y)$. Then the inverse of B, written as $f^{-1}[B]$, is a fuzzy set in X whose membership function is defined by $\mu_{f^{-1}[B]}(x) = \mu_B(f(x))$ for all x in X. Conversely, let A be a fuzzy set in X with membership function $\mu_A(x)$. The image of A, written as $f[A]$, is a fuzzy set in Y whose membership function is given by

$$\mu_{f[A]}(y) = \begin{cases} \sup_{z\in f^{-1}[y]} \{\mu_A(z)\} & \text{if } f^{-1}[y] \text{ is non empty} \\ 0 & \text{other wise.} \end{cases}$$

for all y in Y, where $f^{-1}[y] = \{x \backslash f(x) = y\}$.

Definition 2.4[6]

A fuzzy subset u of fuzzy space X is called fuzzy sp-open (fuzzy sp-closed) set if u ≤ Int cl u ∨ cl Int u (Int cl u ∧ cl Int u ≤ u). The class of all fuzzy sp-open (fuzzy sp-closed) sets in X will be denoted by FSP-O(X) (FSP-C(X)).

Definition 2.5[9]

Let (X, τ) be a topological space and define $\tau(A_\alpha)$ = $\tau_\alpha(say)$ = { $G_\alpha(\neq X)$: $G_\alpha \cap A_\alpha \neq \phi$, for $A_\alpha \in \tau$ and $A_\alpha \neq \phi$, X, for some $\alpha \in J$, where J is the index set }. Define $\tau_f = \{\phi, X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$. The collection τ_f of subsets of X is called the fine collection of subsets of X and (X, τ, τ_f) is said to be the fine space X generated by the topology τ on X.

III. FINE FUZZY ȿp CLOSED SETS IN FINE FUZZY TOPOLOGICAL SPACES

Definition 3.1

Let us consider the fuzzy topological space $(\mathcal{X},\mathcal{T})$ and let $\mathcal{T}(\lambda_\alpha) = \mathcal{T}_\alpha = \{\, \mu_\alpha(\neq 1) \quad \mu_\alpha \wedge \lambda_\alpha \neq 0$, for $\lambda_\alpha \in \mathcal{T}_\alpha$ and $\lambda_\alpha \neq 0, 1$,

Where J is the indexed set for some $\alpha \in J\,\}$. Then the collection $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$ is said to be fine fuzzy topology on $\mathcal{X}$ and $(\mathcal{X},\mathcal{T},\mathcal{T}_f)$ is the fine fuzzy topological space. It is denoted by $\mathcal{F}fTS$.

Definition 3.2

A fuzzy subset λ of a $\mathcal{F}fTS$, is fine fuzzy open set of $\mathcal{X}$ $(FfO(\mathcal{X}))$, if $\lambda \in \mathcal{T}_f$ and its complement is denoted by fine fuzzy closed set $(FfC(\mathcal{X}))$ of $\mathcal{X}$.

Definition 3.3

Let $\mu \in I^{\mathcal{X}}$. Then the fine fuzzy interior of μ is denoted and defined by
$\mathrm{FfInt}(\mu) = \vee\{\, \nu : \mu \geq \nu, \mu$ is a fine fuzzy open set of $\mathcal{X}\,\}$.

Definition 3.4

Let $\nu \in I^{\mathcal{X}}$. Then the fine fuzzy closure of ν is denoted and defined by
$\mathrm{FfCl}(\nu) = \wedge\{\, \mu : \nu \leq \mu, \nu$ is a fine fuzzy closed set of $\mathcal{X}\,\}$

Definition 3.5

Let $\psi : (\mathcal{X},\mathcal{T},\mathcal{T}_f) \to (\mathcal{Y}, \sigma, \sigma_f)$ be a mapping if $\psi^{-1}(\lambda) \in FfO(\mathcal{X})$ for each fine fuzzy open set λ of $\mathcal{Y}$, then ψ is called fine fuzzy- irresolute.

Definition 3.6

A fine fuzzy subset λ of $\mathcal{X}$ is called

1. fine fuzzy pre closed $(FfpC)$ if $\mathrm{FfCl}(\mathrm{FfInt}(\lambda)) \leq \lambda$.
2. fine fuzzy semi closed $(FfsC)$ if $\mathrm{FfInt}(\mathrm{FfCl}(\lambda)) \leq \lambda$.
3. fine fuzzy α -closed $(Ff\alpha C)$ if $\mathrm{FfCl}\left(\mathrm{FfInt}(\mathrm{fFCl}(\lambda))\right) \leq \lambda$.
4. fine fuzzy β-closed $(Ff\beta C)$ if $\mathrm{FfInt}\left(\mathrm{FfCl}(\mathrm{FfInt}(\lambda))\right) \leq \lambda$.
5. fine fuzzy ȿp-closed (FfȿpC) if $\mathrm{FfCl}(\mathrm{FfInt}(\lambda)) \wedge \mathrm{FfInt}(\mathrm{FfCl}(\lambda)) \leq \lambda$.

The class of all fine fuzzy ȿp closed set in $\mathcal{X}$ is denoted by FfȿpC$(\mathcal{X})$ and its complement is fine fuzzy ȿp open set is FfȿpO$(\mathcal{X})$ similarly for all the set

Definition 3.7

Let $(\mathcal{X},\mathcal{T},\mathcal{T}_f)$ be a $\mathcal{F}fTS$. Let λ be fine fuzzy set of $\mathcal{X}$. Then,

1. $FfpInt(\mu) = \vee\{\, \mu : \mu \leq \nu, \mu \in FfpO(\mathcal{X})\,\}$ is fine fuzzy pre Interior.
2. $FfpCl(\nu) = \wedge\{\nu : \nu \geq \mu,\ \nu \in FfpC(\mathcal{X})\,\}$ is fine fuzzy pre closure.
3. $FfsInt(\mu) = \vee\{\mu : \mu \leq \nu, \mu \in FfsO(\mathcal{X})\}$ is fine fuzzy semi Interior.
4. $Ffscl(\nu) = \wedge\{\nu : \nu \geq \mu,\ \nu \in FfsC(\mathcal{X})\,\}$ is fine fuzzy semi closure.
5. $Ff\alpha Int(\mu) = \vee\{\mu : \mu \leq \nu, \mu \in Ff\alpha O(\mathcal{X})\}$ is fine fuzzy α-Interior.
6. $Ff\alpha Cl(\nu) = \wedge\{\nu : \nu \geq \mu,\ \nu \in Ff\alpha C(\mathcal{X})\}$ is fine fuzzy α-closure.
7. $Ff\beta Int(\mu) = \vee\{\mu : \mu \leq \nu, \mu \in Ff\,\beta O(\mathcal{X})\}$ is fine fuzzy β-Interior.
8. $Ff\beta Cl(\nu) = \wedge\{\nu : \nu \geq \mu,\ \nu \in Ff\,\beta C(\mathcal{X})\}$ is fine fuzzy β-closure.
9. Ffȿ$pInt(\mu) = \vee\{\mu : \mu \leq \nu, \mu \in$ FfȿpO$(\mathcal{X})$ is fine fuzzy ȿp-Interior.
10. Ffȿ$pCl(\nu) = \wedge\{\,\nu : \nu \geq \mu,\ \nu \in$ FfȿpC$(\mathcal{X})\}$ is fine fuzzy ȿp-closure.

Remark 3.1 A fine fuzzy set λ of $\mathcal{X}$. Then

(i) $\mathrm{FfsCl}(\lambda) \geq \lambda$ and $\mathrm{FfsInt}(\lambda) \leq \lambda$,

(ii) $\lambda \leq \mu \Longrightarrow \mathrm{FfsCl}(\lambda) \leq \mathrm{FfsCl}(\mu)$, $FfsInt(\lambda) \leq FfsInt(\mu)$.

(iii) $\mathrm{FfInt}(\mathrm{FfCl}(\lambda)) \leq \mathrm{FfsCl}(\lambda)$, $\mathrm{Ffcl}(\mathrm{FfInt}(\lambda)) \geq \lambda$.

Proposition 3.1

A fine fuzzy set λ of $\mathcal{X}$, then following properties are true:

(i) $\mathrm{FfsCl}(\lambda) \geq \lambda \vee \mathrm{FfInt}(\mathrm{FfCl}(\lambda))$ and $\mathrm{FfsInt}(\lambda) \leq \lambda \wedge \mathrm{Ffcl}(\mathrm{FfInt}(\lambda))$.

(ii) $\mathrm{FfpCl}(\lambda) \geq \lambda \vee \mathrm{Ffcl}(\mathrm{FfInt}(\lambda))$ and $\mathrm{FfpInt}(\lambda) \leq \lambda \wedge \mathrm{FfInt}(\mathrm{FfCl}(\lambda))$.

Proof

(i) Since, by the above remark it is easy.

(ii) Since, $\lambda \leq \mathrm{FfpCl}(\lambda)$, $\mathrm{FfpCl}(\lambda)$ is a fine fuzzy pre closed set, $\mathrm{Ffcl}(\mathrm{FfInt}(\lambda)) \leq \mathrm{FfCl}(\mathrm{FfInt}(\mathrm{FfpCl}(\lambda))) \leq \mathrm{FfpCl}(\lambda)$.Thus, $\lambda \vee \mathrm{FfCl}(\mathrm{FfInt}(\lambda)) \leq \mathrm{FfpCl}(\lambda)$.Since, $\lambda \geq \mathrm{FfpInt}(\lambda)$, $\mathrm{FfpInt}(\lambda)$ is a fine fuzzy preopen set, $\mathrm{FfInt}(\mathrm{FfCl}(\lambda)) \geq \mathrm{FfInt}(\mathrm{FfCl}(\mathrm{FfpInt}(\lambda))) \geq \mathrm{FfpInt}(\lambda)$Thus, $\lambda \wedge \mathrm{FfInt}(\mathrm{FfCl}(\lambda)) \geq \mathrm{FfpInt}(\lambda)$.

Proposition 3.2

For any fine fuzzy subset λ of $\mathcal{X}$, then

(i) λ is fine fuzzy sp-closed is equivalent to

(ii) $\lambda \geq \mathrm{FfpCl}(\lambda) \wedge \mathrm{FfsCl}(\lambda)$

Proof (i) ⇒ (ii)

Given that λ is a fine fuzzy ȿp-closed,

(i.e) $\lambda \geq \mathrm{FfInt}(\mathrm{FfCl}(\lambda)) \wedge \mathrm{FfCl}(\mathrm{FfInt}(\lambda))$.

Then $\mathrm{FfpCl}(\lambda) \wedge \mathrm{FfsCl}(\lambda) \geq \lambda \vee \mathrm{FfCl}(\mathrm{FfInt}(\lambda)) \wedge \lambda \vee \mathrm{FfInt}(\mathrm{FfCl}(\lambda))$.

$\geq \lambda \vee [\mathrm{FfCl}(\mathrm{FfInt}(\lambda)) \wedge \mathrm{FfInt}(\mathrm{FfCl}(\lambda))]$.

$\geq \lambda \vee \lambda = \lambda$.

Hence (i) ⇒ (ii)

Conversely, (ii)⇒(i). Assume that $\lambda \leq \mathrm{FfpInt}(\lambda) \vee \mathrm{FfsInt}(\lambda)$. Since, by Definition 3.6 we get λ is fine fuzzy ȿp-closed.

Proposition 3.3

For any two fine fuzzy sets λ and μ of $\mathcal{X}$. Hence the following are true.

(i) FfspCl(λ) is fine fuzzy sp closed.
(ii) $\lambda \subseteq$ FfspC($\mathcal{X}$) $\Leftrightarrow \lambda =$ FfspCl(λ).
(iii) $\lambda \leq \mu \Rightarrow$ FfspCl(λ) $\leq$ FfspCl(μ).
(iv) FfspInt(λ) is fine fuzzy sp open.
(v) $\lambda \subseteq$ FfspO($\mathcal{X}$) $\Leftrightarrow \lambda =$ FfspInt(λ).
(vi) $\lambda \leq \mu \Rightarrow$ FfspInt(λ) $\leq$ FfspInt(μ).
(vii) FfInt(λ) $\leq$ FfspInt(λ) $\leq \lambda \leq$ FfspCl(λ) $\leq$ FfCl(λ).

Proof

(i) Assume that μ is FfspCl(λ) by the Definition 3.4,
μ = FfspCl(λ) = ∧ { η : $\eta \geq \lambda$, η ∈ FfspC($\mathcal{X}$) }.
Hence, μ is fine fuzzy sp closed.

(ii) Assume that $\lambda =$ FfspCl(λ) then by the Definition 3.4 we have,
$\lambda =$ FfspCl(λ) = ∧ { μ: μ ≥ λ, μ ∈ FfspC($\mathcal{X}$) }
⇔ λ ∈ ∧ { μ : μ ≥ λ, μ ∈ FfspC($\mathcal{X}$) }.
⇔ λ is FfspC($\mathcal{X}$).

(iii) Assume that $\lambda \leq \mu$ then FfspCl(λ) ≤ FfspCl(μ).

(iv) Assume that λ is FfspInt(λ) by Definition 3.3 we get
FfspInt(λ) = ∨ { μ: μ ≤ λ, μ ∈ Ffsp$O$($\mathcal{X}$) }.
Hence λ is fine fuzzy sp - open.

(v) Assume that $\lambda =$ Ff spInt(λ) then by Definition 3.3 we have,
Ff spInt(λ) = ∨ {μ: μ ≤ λ, μ ∈ Ffsp$O$($\mathcal{X}$) }.
⇔ λ ∈ ∨{ μ: μ ≤ λ, μ ∈ FfspO($\mathcal{X}$) }
⇔ λ ⊆ FfspO($\mathcal{X}$).

(vi) Assume that $\lambda \leq \mu$ then FfspInt(λ) ≤ FfspInt(μ).

(vii) Proof is immediate from Definition 3.6

Lemma 3.1

A fine fuzzy set $\lambda \in \mathcal{F}fTS(\mathcal{X})$. Then

(i) FfCl($1_{\mathcal{X}} - \lambda$) = $1_{\mathcal{X}}$ – FfInt (λ) and
(ii) FfInt($1_{\mathcal{X}} - \lambda$) = $1_{\mathcal{X}}$– FfCl(λ).

Proof

(i) Let λ be fine fuzzy set and μ be fine fuzzy open set with $\mu \leq \lambda$. Let ν ≥ $1_{\mathcal{X}} - \lambda$ be fine fuzzy closed set. Then
FfInt(λ) = ∨ {$1_{\mathcal{X}}$ − ν: ν ∈ FfspC($\mathcal{X}$) and ν ≥ $1_{\mathcal{X}} - \lambda$ }
= $1_{\mathcal{X}}$ − ∧ {ν: ν ∈ fine fuzzy closed set and ν ≥ $1_{\mathcal{X}} - \lambda$}
FfInt(λ) = $1_{\mathcal{X}}$ – FfCl($1_{\mathcal{X}} - \lambda$).
Thus, FfCl($1_{\mathcal{X}} - \lambda$) = $1_{\mathcal{X}}$ – FfInt(λ).

(ii) Let μ be a fine fuzzy set and λ be fine fuzzy closed with $\lambda \leq \mu$. Hence, for a fine fuzzy open set ν ≤ $1_{\mathcal{X}} - \lambda$.
FfCl(λ) = ∧ {$1_{\mathcal{X}} - \mu$: ν ∈ Ffsp$O$($\mathcal{X}$) ν ≤ $1_{\mathcal{X}} - \lambda$}
= $1_{\mathcal{X}}$ − ∨ { μ: μ is a fine fuzzy open set and ν ≤ $1_{\mathcal{X}} - \lambda$}
FfCl(λ) = $1_{\mathcal{X}}$ – FfInt($1_{\mathcal{X}} - \lambda$).Thus, FfCl($1_{\mathcal{X}} - \lambda$) = $1_{\mathcal{X}}$ – FfInt(λ).

Lemma 3.2

A fine fuzzy set $\lambda \in \mathcal{F}fTS(\mathcal{X})$,

(i) FfspCl($1_{\mathcal{X}} - \lambda$) = $1_{\mathcal{X}}$ – FfspInt (λ) and
(ii) FfspInt($1_{\mathcal{X}} - \lambda$) = $1_{\mathcal{X}}$ – FfspCl(λ).

Proof

(i) . A fine fuzzy set λ, μ is fine fuzzy sp open set with $\mu \leq \lambda$. Let ν ≥ $1_{\mathcal{X}} - \lambda$ be fine fuzzy sp closed set. Then
FfspInt(λ) = ∨ {$1_{\mathcal{X}}$ − ν: ν ∈ FfspC($\mathcal{X}$) and ν ≥ $1_{\mathcal{X}} - \lambda$ }
= $1_{\mathcal{X}}$ −∧ {ν: ν ∈ FfspC($\mathcal{X}$) and ν ≥ $1_{\mathcal{X}} - \lambda$ }.
FfspInt(λ) = $1_{\mathcal{X}}$ – FfspCl($1_{\mathcal{X}} - \lambda$).
Thus, FfspCl($1_{\mathcal{X}} - \lambda$) = $1_{\mathcal{X}}$ – FfspInt(λ).

(ii). A fine fuzzy set μ, λ ∈ fine fuzzy sp closed with $\lambda \leq \mu$. Then for a fine fuzzy sp open set ν ≤ $1_{\mathcal{X}} - \lambda$.
FfspCl (λ) = ∧ {$1_{\mathcal{X}} - \mu$: ν ∈ FfspO($\mathcal{X}$) , ν ≤ $1_{\mathcal{X}} - \lambda$}
= $1_{\mathcal{X}}$ − ∨ { ν ∈ FfspO($\mathcal{X}$), ν ≤ $1_{\mathcal{X}} - \lambda$}
FfspCl(λ) = $1_{\mathcal{X}}$ – FfspInt($1_{\mathcal{X}} - \lambda$).
Thus, FfspInt($1_{\mathcal{X}} - \lambda$) = $1_{\mathcal{X}}$ – FfspCl(λ).

Proposition 3.4

A fine fuzzy set λ of $\mathcal{X}$. Thus the following statements are hold

(i) FfspCl(λ) ≥ λ ∨ ((FfInt(FfCl(λ)) ∧ (FfCl(FfInt(λ)))).
(ii) FfspInt(λ) ≤ λ ∧ ((FfInt(FfCl(λ)) ∨ (FfCl(FfInt(λ)))).
(iii) FfspCl(λ) ≥ (FfsCl(λ)) ∧ (Ffpcl(λ)).
(iv) FfspInt(λ) ≤ (FfsInt(λ)) ∨ (FfpInt(λ)).

Proof

(i) Since, Ffspcl(λ) is fine fuzzy sp closed set (i.e)
λ ≥ FfInt(FfCl(λ)) ∧ (FfCl(FfInt(λ))) and λ ≤ FfspCl(λ).
FfspCl(λ) ≥ λ ≥ FfInt(FfCl(λ)) ∧ (FfCl(FfInt(λ))).
Hence, FfspCl(λ) ≥ λ ∨ ((FfInt(FfCl(λ)) ∧ (FfCl(FfInt(λ)))).

(ii) Since, FfspInt(λ) is fine fuzzy sp open set (i.e)
λ ≤ FfInt(FfCl(λ)) ∨ (FfCl(FfInt(λ))) and λ ≤ FfspInt(λ).
FfspInt(λ) ≤ λ ≤ FfInt(FfCl(λ)) ∨ (FfCl(FfInt(λ))).
Hence, FfspInt(λ) ≤ λ ∧ ((FfInt(FfCl(λ)) (FfCl(FfInt(λ)))).

(iii) Assume

(FfsCl(λ)) ∧ (FfpCl(λ)) ≥ (λ ∨ FfInt(FfsCl(λ))) ∧ (λ ∨ (FfCl(FfInt(λ))))

= λ ∨ ((FfInt(FfCl(λ)) ∧ (FfCl(FfInt(λ))))

= (FfInt(FfCl(λ)) ∧ (FfCl(FfInt(λ)))) ≥

Thus, (FfsCl(λ)) ∧ (FfpCl(λ)) ≥ FfspCl(λ).

(iv) (FfsInt(λ)) ∨ (FfpInt(λ)) ≤ (λ ∧ FfCl(FfInt(λ))) ∨ (λ ∧ FfInt(FfCl(λ))) = λ ∧ (((FfCl(FfInt(λ))) ∨ FfInt(FfCl(λ)))

$= \left(\left(\mathrm{FfCl}(\mathrm{FfInt}(\lambda))\right) \vee \mathrm{FfInt}(\mathrm{FfCl}(\lambda))\right) \leq \mathrm{Ff}\mathfrak{sp}\mathrm{int}(\lambda).$

Proposition 3.5

Let λ be a fine fuzzy subset of $\mathcal{X}$. Then the equivalent statements are valid

(i) $\lambda \in$ fine fuzzy 𝔰𝔭 closed.

(ii) $\lambda^c \in$ fine fuzzy 𝔰𝔭 open.

(iii) $\lambda \geq ((FfInt(FfCl(\lambda)) \wedge (FfCl(FfInt(\lambda))))$.

(iv) $\lambda^c \leq ((FfInt(FfCl(\lambda)) \vee (FfCl(FfInt(\lambda))))$.

Proof (i) ⇔ (ii) follows from Lemma 3.2

(i) ⇒ (iii)

By definition ∃ a fine fuzzy 𝔰𝔭 closed set :

$\lambda \geq ((FfInt(FfCl(\lambda)) \wedge (FfCl(FfInt(\lambda))))$.

(i) ⇒ (iv)

By definition ∃a fine fuzzy 𝔰𝔭 closed set :

$\lambda \geq ((FfInt(FfCl(\lambda)) \wedge (FfCl(FfInt(\lambda))))$

and $1_X - \lambda$ is fine fuzzy 𝔰𝔭 open set. Hence, $\lambda^c \leq ((FfInt(FfCl(\lambda)) \vee (FfCl(FfInt(\lambda))))$.

Proposition 3.6

A fine fuzzy set $\lambda \in \mathcal{F}fTS(\mathcal{X})$. Hence, the properties are hold

(i) $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(0_X) = 0_X$.

(ii) $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda)) = \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda)$.

(iii) $\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda)) = \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda)$.

Proof

It is obvious.

Proposition 3.7

A $\mathcal{F}fTS(\mathcal{X})$, the relations are valid

(i) $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda \vee \mu) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda) \vee \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\mu)$

(ii) $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda \wedge \mu) \leq \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda) \wedge \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\mu)$.

(iii) $\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda \vee \mu) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda) \vee \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\mu)$.

(iv) $\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda \wedge \mu) \leq \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda) \wedge \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\mu)$

Proof

(i) $\lambda \leq \lambda \vee \mu$ or $\mu \leq \lambda \vee \mu$ that implies $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda) \leq \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda \vee \mu)$ or $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\mu) \leq \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda \vee \mu)$. Therefore, $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda \vee \mu) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda) \vee \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\mu)$.

(ii) $\lambda \geq \lambda \wedge \mu$ or $\mu \geq \lambda \wedge \mu$ that implies $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda \wedge \mu)$ or $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\mu) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda \wedge \mu)$. Therefore, $\mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda \wedge \mu) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\lambda) \wedge \mathrm{Ff}\mathfrak{sp}\mathrm{Cl}(\mu)$.

(iii) $\lambda \leq \lambda \vee \mu$ or $\mu \leq \lambda \vee \mu$ that implies $\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda) \leq \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda \vee \mu)$ or $\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\mu) \leq \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda \vee \mu)$. Therefore, $\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda \vee \mu) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda) \vee \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\mu)$.

(iv) $\lambda \geq \lambda \wedge \mu$ or $\mu \geq \lambda \wedge \mu$ that implies $\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda \wedge \mu)$ or $\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\mu) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda \wedge \mu)$. Therefore, $\mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda \wedge \mu) \geq \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\lambda) \wedge \mathrm{Ff}\mathfrak{sp}\mathrm{Int}(\mu)$.

IV. RESULT DESCRIPTIONS

Proposition 4.1

Every FfC is $FfsC$.

Proof

For a fine fuzzy closed set λ, $\mathrm{FfCl}(\lambda) = \lambda$.
To prove λ is fine fuzzy semi closed
Since, $\mathrm{Ff}Cl(\lambda) \leq \lambda$ and $\mathrm{FfInt}(\lambda) \leq \lambda$
$FfInt((FfCl(\lambda))) \leq FfInt(\lambda)$.
$FfInt(fFCl(\lambda)) \leq \lambda$.
Hence, every FfC set is $FfsC$.

Note 4.1

The converse of Proposition 4.1 need not be true, shows in the following

Example 4.1

Let $\mathcal{X}$ = {a, b} be a nonempty set. Let λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 $\in I^X$ be defined as $\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.5$; $\lambda_2(a) = 0.6$, $\lambda_2(b) = 0.5$; $\lambda_3(a) = 0.4$, $\lambda_3(b) = 0.4$; $\lambda_4(a) = 0.3$, $\lambda_4(b) = 0.4$; $\lambda_5(a) = 0.5$, $\lambda_5(b) = 0.6$; $\lambda_6(a) = 0.5$, $\lambda_6(b) = 0.5$. Let $\mathcal{T} = \{0, 1, \lambda_1, \lambda_2\}$ be the fuzzy topology on $\mathcal{X}$ and Then $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$ fine fuzzy topology defined as $\mathcal{T}(\lambda_\alpha) = \mathcal{T}_\alpha$ $\mathcal{T}(\lambda_1) = \mathcal{T}_1 = \lambda_3 = \{\lambda_3 \wedge \lambda_1 = \lambda_4 \neq 0\}$, $\mathcal{T}(\lambda_2) = \mathcal{T}_1 = \lambda_5 = \{\lambda_5 \wedge \lambda_2 = \lambda_6 \neq 0\}$. Thus, $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ is $\mathcal{F}fTS$. Let $\lambda_3 \in FfsC$, but $\lambda_3 \notin FfC$.

Proposition 4.2

Every FfC is $Ff\alpha C$.

Proof

Let λ be fine fuzzy closed, $\mathrm{FfCl}(\lambda) = \lambda$.
To prove: $\lambda \in Ff\alpha C$.
Since, $\lambda \leq \mathrm{Ff}Cl(\lambda)$
$FfInt(\lambda) \leq (FfInt(FfCl(\lambda)))$
$FfCl(FfInt(\lambda)) \leq FfCl\,(FfInt(FfCl(\lambda)))$
$FfInt(\lambda) \leq FfCl\,(FfInt(FfCl(\lambda)))$
$\lambda \leq FfCl\,(FfInt(FfCl(\lambda)))$.
Hence, every FfC is $Ff\alpha C$.

Note 4.2

The converse of Proposition 4.2 need not be true shows in the following

Example 4.2

Let $\mathcal{X}$ = {a, b} be a nonempty set. Let λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 $\in I^X$ be defined as $\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.5$; $\lambda_2(a) = 0.6$, $\lambda_2(b) = 0.5$; $\lambda_3(a) = 0.4$, $\lambda_3(b) = 0.4$; $\lambda_4(a) = 0.3$, $\lambda_4(b) = 0.4$; $\lambda_5(a) = 0.5$, $\lambda_5(b) = 0.6$; $\lambda_6(a) = 0.5$, $\lambda_6(b) = 0.5$. Then $(\mathcal{X}, \mathcal{T})$ is fTS where $\mathcal{T} = \{0, 1, \lambda_1, \lambda_2\}$ and $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$ a fine fuzzy topology defined as $\mathcal{T}(\lambda_1) = \mathcal{T}_1 = \lambda_3 = \{\lambda_3 \wedge \lambda_1 = \lambda_4 \neq$

$0\}$, $\mathcal{T}(\lambda_2) = \mathcal{T}_2 = \lambda_5 = \{\lambda_5 \wedge \lambda_2 = \lambda_6 \neq 0\}$. Thus, $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ is $\mathcal{FfTS}$. Let $\lambda_3 \in Ff\alpha C$, but $\lambda_3 \notin FfC$.

Proposition 4.3

Every $FfsC$ is $Ff\mathfrak{sp}C$.

Proof

Let $\lambda \in FfsC$.

Since, $FfInt(FfCl(\lambda)) \leq \lambda \quad \rightarrow \quad (1)$

Since, $FfInt(\lambda) \leq \lambda$

$FfCl(FfInt(\lambda)) \leq Ffcl(\lambda) \quad \rightarrow \quad (2)$

$FfCl(FfInt(\lambda)) \leq \lambda$

From (1) and (2)

$\lambda \wedge FfCl(\lambda) \geq FfInt(FfCl(\lambda)) \wedge FfCl(FfInt(\lambda))$

$\lambda \geq FfInt(FfCl(\lambda)) \wedge FfCl(FfInt(\lambda))$.

Hence, every $FfsC$ is $Ff\mathfrak{sp}C$.

Note 4.3

The converse of Proposition 4.3 need not be true shows in the following

Example 4.3

Let $\mathcal{X} = \{a, b\}$ be a nonempty set. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in I^X$ be defined as $\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.5$; $\lambda_2(a) = 0.6$, $\lambda_2(b) = 0.5$; $\lambda_3(a) = 0.4$, $\lambda_3(b) = 0.4$; $\lambda_4(a) = 0.3$, $\lambda_4(b) = 0.4$; $\lambda_5(a) = 0.5$, $\lambda_5(b) = 0.6$; $\lambda_6(a) = 0.5$, $\lambda_6(b) = 0.5$. Let $\mathcal{T} = \{0, 1, \lambda_1, \lambda_2\}$ be the fuzzy topology on X and Then $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$ be fine fuzzy topology defined as $\mathcal{T}(\lambda_1) = \mathcal{T}_1 = \lambda_3 = \{\lambda_3 \wedge \lambda_1 = \lambda_4 \neq 0\}$, $\mathcal{T}(\lambda_2) = \mathcal{T}_2 = \lambda_5 = \{\lambda_5 \wedge \lambda_2 = \lambda_6 \neq 0\}$. Thus, $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ is $\mathcal{FfTS}$. Let $\lambda_6 \in Ff\mathfrak{sp}C$ but $\lambda_6 \notin FfsC$.

Proposition 4.4

Every $FfpC$ is $Ff\mathfrak{sp}C$.

Proof

Let λ be fine fuzzy pre-closed set.

Since, $\lambda \geq FfCl(FfInt(\lambda))$ and $\lambda \geq FfInt(FfCl(\lambda))$.

$\Rightarrow \lambda \geq FfCl(FfInt(\lambda)) \wedge FfInt(FfCl(\lambda))$.

Hence, every $FfpC$ is $Ff\mathfrak{sp}C$.

Note 4.4

The converse of Proposition 4.4 need not be true shows in the following

Example 4.4

Let $\mathcal{X} = \{a, b\}$ be a nonempty set. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in I^X$ be defined as $\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.5$; $\lambda_2(a) = 0.6$, $\lambda_2(b) = 0.5$; $\lambda_3(a) = 0.4$, $\lambda_3(b) = 0.4$; $\lambda_4(a) = 0.3$, $\lambda_4(b) = 0.4$; $\lambda_5(a) = 0.5$, $\lambda_5(b) = 0.6$; $\lambda_6(a) = 0.5$, $\lambda_6(b) = 0.5$. Let $\mathcal{T} = \{0, 1, \lambda_1, \lambda_2\}$ be the fuzzy topology on $\mathcal{X}$ and Then $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$ be fine fuzzy topology defined as $\mathcal{T}(\lambda_1) = \mathcal{T}_1 = \lambda_3 = \{\lambda_3 \wedge \lambda_1 = \lambda_4 \neq 0\}$, $\mathcal{T}(\lambda_2) = \mathcal{T}_2 = \lambda_5 = \{\lambda_5 \wedge \lambda_2 = \lambda_6 \neq 0\}$. Thus, $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ is $\mathcal{FfTS}$. Let $\lambda_5 \in Ff\mathfrak{sp}C$, but $\lambda_5 \notin FfpC$.

Proposition 4.5

Every Ff$\mathfrak{sp}$C is $Ff\beta C$.

Proof

Let λ be fine fuzzy $\mathfrak{sp}$-closed set. i.e $\lambda \geq FfCl(FfInt(\lambda)) \wedge FfInt(FfCl(\lambda))$.

To Prove $\lambda \geq FfInt(FfCl(FfInt(\lambda)))$.

Also, $\lambda \geq FfCl(FfInt(\lambda))$ and $\lambda \geq FfInt(FfCl(\lambda))$.

$\lambda \geq FfInt(\lambda) \geq FfInt(FfCl(FfInt(\lambda)))$.

Hence, every Ff$\mathfrak{sp}$C is $Ff\beta C$.

Note 4.5

The converse of Proposition 4.5 need not be true shows in the following

Example 4.5

Let $\mathcal{X} = \{a, b\}$ be a nonempty set. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in I^X$ be defined as $\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.5$; $\lambda_2(a) = 0.6$, $\lambda_2(b) = 0.5$; $\lambda_3(a) = 0.4$, $\lambda_3(b) = 0.4$; $\lambda_4(a) = 0.3$, $\lambda_4(b) = 0.4$; $\lambda_5(a) = 0.5$, $\lambda_5(b) = 0.6$; $\lambda_6(a) = 0.5$, $\lambda_6(b) = 0.5$; $\lambda_7(a) = 0.6$, $\lambda_7(b) = 0.7$. Let $\mathcal{T} = \{0, 1, \lambda_1, \lambda_2\}$ be the fuzzy topology on $\mathcal{X}$ and Then $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$ be fine fuzzy topology defined as $\mathcal{T}(\lambda_1) = \mathcal{T}_1 = \lambda_3 = \{\lambda_3 \wedge \lambda_1 = \lambda_4 \neq 0\}$, $\mathcal{T}(\lambda_2) = \mathcal{T}_2 = \lambda_5 = \{\lambda_5 \wedge \lambda_2 = \lambda_6 \neq 0\}$. Thus, $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ is $\mathcal{FfTS}$. Let $\lambda_7 \in Ff\beta C$, but $\lambda_7 \notin Ff\mathfrak{sp}C$.

Proposition 4.6

Every $FfpC$ is $Ff\beta C$.

Proof

Let $\lambda \in FfpC$.

Since, $FfCl(FfInt(\lambda)) \leq \lambda$

To Prove $FfInt(FfCl(FfInt(\lambda))) \leq \lambda$.

$\Rightarrow FfInt(FfCl(FfInt(\lambda))) \leq FfInt(\lambda) \leq \lambda$.

Hence, every $FfpC$ is $Ff\beta C$.

Note 4.6

The converse of Proposition 4.6 need not be true shows in the following

Example 4.6

Let $\mathcal{X} = \{a, b\}$ be a nonempty set. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in I^X$ be defined as $\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.5$; $\lambda_2(a) = 0.6$, $\lambda_2(b) = 0.5$; $\lambda_3(a) = 0.4$, $\lambda_3(b) = 0.4$; $\lambda_4(a) = 0.3$, $\lambda_4(b) = 0.4$; $\lambda_5(a) = 0.5$, $\lambda_5(b) = 0.6$; $\lambda_6(a) = 0.5$, $\lambda_6(b) = 0.5$; $\lambda_7(a) = 0.5$, $\lambda_7(b) = 1$. Let $T = \{0, 1, \lambda_1, \lambda_2\}$ be the fuzzy topology on $\mathcal{X}$ and Then $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}$ be fine fuzzy topology defined as $\mathcal{T}(\lambda_1) = \mathcal{T}_1 = \lambda_3 = \{\lambda_3 \wedge \lambda_1 = \lambda_4 \neq 0\}$, $\mathcal{T}(\lambda_2) = \mathcal{T}_2 = \lambda_5 = \{\lambda_5 \wedge \lambda_2 = \lambda_6 \neq 0\}$. Thus, $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ is $\mathcal{FfTS}$. Let $\lambda_7 \in Ff\beta C$, but $\lambda_7 \notin FfpC$.

Proposition 4.7

Every FfC is $Ff\mathfrak{sp}C$.

Proof

Let $\lambda \in FfC(\mathcal{X})$ and by Proposition 3.9 & 3.12 we have

Since, $FfInt(\lambda) \leq \lambda$ and $FfCl(FfInt(\lambda)) \leq (FfCl(\lambda))$

$$FfCl(FfInt(\lambda)) \leq (FfCl(\lambda)) \leq \lambda \qquad (1)$$

Since, $\lambda \geq FfCl(\lambda)$, $FfInt(\lambda) \geq FfInt(FfCl(\lambda))$

$$\lambda \geq FfInt(FfCl(\lambda)) \qquad (2)$$

Hence, from (1) and (2) $\lambda \in Ff\mathfrak{sp}C$.

Note 4.7

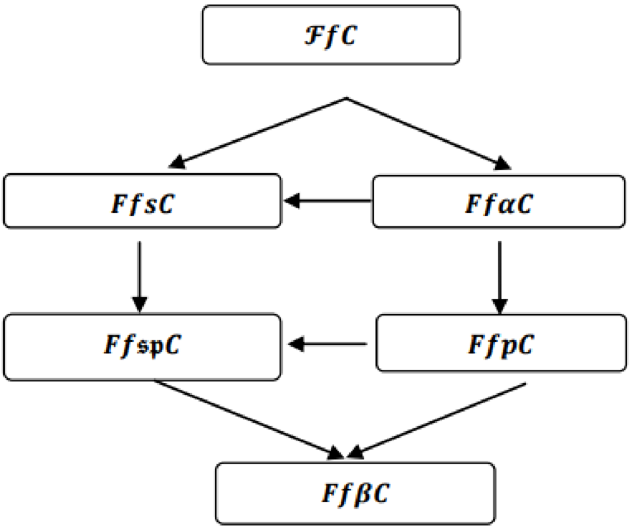
The converse of Proposition 4.7 need not be true shows in the following

Example 4.7

Let $\mathcal{X}$ = {a, b} be a nonempty set. Let λ_1, λ_2, λ_3, λ_4, λ_5, $\lambda_6 \in I^{\mathcal{X}}$ be defined as $\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.5$; $\lambda_2(a) = 0.6$, $\lambda_2(b) = 0.5$; $\lambda_3(a) = 0.4$, $\lambda_3(b) = 0.4$; $\lambda_4(a) = 0.3$, $\lambda_4(b) = 0.4$; $\lambda_5(a) = 0.5$, $\lambda_5(b) = 0.6$; $\lambda_6(a) = 0.5$, $\lambda_6(b) = 0.5$. Let $\mathcal{T} = \{0, 1, \lambda_1, \lambda_2\}$ be the fuzzy topology on $\mathcal{X}$ and Then $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$ be fine fuzzy topology defined as $\mathcal{T}(\lambda_1) = \mathcal{T}_1 = \lambda_3 = \{\lambda_3 \wedge \lambda_1 = \lambda_4 \neq 0\}$, $\mathcal{T}(\lambda_2) = \mathcal{T}_2 = \lambda_5 = \{\lambda_5 \wedge \lambda_2 = \lambda_6 \neq 0\}$. Thus, $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ is $\mathcal{F}fTS$. Let $\lambda_5 \in Ff\mathfrak{sp}C$, but $\lambda_5 \notin FfC$.

Note 4.8

Clearly the above discussions gives the following implications:



(figure - 4.1)

Interrelations between $Ff\mathfrak{sp}C$ and other types of fine fuzzy sets

Proposition 4.8

Let $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ be $\mathcal{F}fTS$. Then

(i) An arbitrary union of fine fuzzy $\mathfrak{sp}$ open sets are fine fuzzy open set.

(ii) A finite intersection of fine fuzzy $\mathfrak{sp}$-closed sets are not fine fuzzy closed.

V. FINE FUZZY $\mathfrak{sp}$ CONTINUOUS FUNCTION

Definition 5.1

A function ψ: $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f) \rightarrow (\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$ is

a. $Ff\mathfrak{sp}O$ ($Ff\mathfrak{sp}C$) if $\psi^{-1}(v) \in Ff\mathfrak{sp}O(\mathcal{Y})$ $[Ff\mathfrak{sp}C(\mathcal{Y})]$ $\forall v \in FfO(\mathcal{X})$ $[FfC(\mathcal{X})]$.

b. $Ff\mathfrak{sp}^*O$ ($Ff\mathfrak{sp}^*C$) if $\psi(v) \in Ff\mathfrak{sp}O(\mathcal{Y})$ $[Ff\mathfrak{sp}C(\mathcal{Y})]$ $\forall v \in FfO(\mathcal{X})$ $[FfC(\mathcal{X})]$.

c. $Ff\mathfrak{sp}^{**}O$($Ff\mathfrak{sp}^{**}C$)if $\psi(v) \in FfO(\mathcal{X})$ $[FfC(\mathcal{X})]$ $\forall v \in Ff\mathfrak{sp}O(\mathcal{Y})$ $[Ff\mathfrak{sp}C(\mathcal{Y})]$.

Definition 5.2

Let $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$and $(\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$be two$\mathcal{F}fTSs$. Then a map $\psi: (\mathcal{X}, \mathcal{T}, \mathcal{T}_f) \rightarrow (\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$ is

a) Fine fuzzy continuous (in short Ffcontinuous) if $\psi^{-1}(v) \in FfC(\mathcal{X})[FfO(\mathcal{X})]$ $\forall v \in FfC(\mathcal{Y})$ $[FfO(\mathcal{Y})]$

b) Fine fuzzy $\mathfrak{sp}$continuous (Ff $\mathfrak{sp}$continuous) if $\psi^{-1}(v) \in Ff\mathfrak{sp}C(\mathcal{X})[Ff\mathfrak{sp}O(\mathcal{X})]$ $\forall v \in FfC(\mathcal{Y})$ $[FfO(\mathcal{Y})]$.

c) Fine fuzzy $\mathfrak{sp}$*continuous (Ff $\mathfrak{sp}$*continuous) if $\psi^{-1}(v) \in Ff\mathfrak{sp}C(\mathcal{X})[Ff\mathfrak{sp}O(\mathcal{X})]$ $\forall v \in Ff\mathfrak{sp}C(\mathcal{Y})$ $[Ff\mathfrak{sp}O(\mathcal{Y})]$.

d) Fine fuzzy $\mathfrak{sp}$**continuous (Ff $\mathfrak{sp}$**continuous) if $\psi^{-1}(v) \in FfC(\mathcal{X})[FfO(\mathcal{X})]$ $\forall v \in Ff\mathfrak{sp}C(\mathcal{Y})$ $[Ff\mathfrak{sp}O(\mathcal{Y})]$.

Proposition 5.1

Every Ff $\mathfrak{sp}$**continuous is $Ff\mathfrak{sp}$*continuous.

Proof

Assume two $\mathcal{F}fTSs$ $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$and $(\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$. A map $\psi: (\mathcal{X}, \mathcal{T}, \mathcal{T}_f) \rightarrow (\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$ is $Ff\mathfrak{sp}$**continuous. Then $\forall$ $v \in Ff\mathfrak{sp}C(\mathcal{Y})$, $\psi^{-1}(v) \in FfC(\mathcal{X})$. By Proposition 4.7, every FfC is $Ff\mathfrak{sp}C$. $\therefore$ $\psi^{-1}(v) \in Ff\mathfrak{sp}C(\mathcal{X})$, $\forall$ $v \in Ff\mathfrak{sp}C(\mathcal{Y})$. Hence, ψ is Ff $\mathfrak{sp}$*continuous function.

Note 5.1

The converse of Proposition 5.1 need not be true shows in the following

Example 5.1

Let $\mathcal{X}$ = {a, b} be a non empty set. Let ζ_1, $\zeta_2 \in I^{\mathcal{X}}$ be defined by $\zeta_1(a) = 0.3$, $\zeta_1(b) = 0.4$; $\zeta_2(a) = 0.4$, $\zeta_2(b) = 0.6$. Let $\mathcal{T} = \{0, 1, \zeta_1, \zeta_2\}$ be the fuzzy topology on $\mathcal{X}$ and $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$ be fine fuzzy topology on $\mathcal{X}$, defined as $\mathcal{T}(\zeta_1) = \mathcal{T}_1 = \zeta_1 = \{\zeta_1 \wedge \zeta_1 = \zeta_1 \neq 0\}$, $\mathcal{T}(\zeta_2) = \mathcal{T}_2 = \zeta_2 = \{\zeta_2 \wedge \zeta_2 = \zeta_2 \neq 0\}$. Thus, $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$is a $\mathcal{F}fTS$. Let ψ: $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f) \rightarrow (\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ be an identity map. Then 'ψ' is $\mathcal{F}f\mathfrak{sp}$* continuous but not $\mathcal{F}f\mathfrak{sp}$** continuous. Since, $\vartheta_2(a) = 0.5$, $\vartheta_2(b) = 0.6$ $\in$ $Ff\mathfrak{sp}C(\mathcal{X})$, but $\psi^{-1}(\vartheta_2) = \vartheta_2 \notin FfC(\mathcal{X})$. Thus, ψ is not $Ff\mathfrak{sp}$**continuous. $\therefore$, every Ff $\mathfrak{sp}$*continuous need not be Ff $\mathfrak{sp}$**continuous.

Proposition 5.2

Every $Ff\mathfrak{sp}$*continuous is $Ff\mathfrak{sp}$continuous.

Proof

Consider two $\mathcal{F}f\mathcal{T}\mathcal{S}s$ $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)and(\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$. A map $\psi : (\mathcal{X},\mathcal{T},\mathcal{T}_f) \rightarrow (\mathcal{Y},\mathcal{S},\mathcal{S}_f)$ is fine fuzzy $\mathfrak{sp}$*continuous. Then, $\forall\, v \in Ff\mathfrak{sp}C(\mathcal{Y})$, $\psi^{-1}(v) \in Ff\mathfrak{sp}C(\mathcal{X})$. By Proposition 4.7, every FfC is $Ff\mathfrak{sp}C$. $\therefore$, $\psi^{-1}(v) \in Ff\mathfrak{sp}C(\mathcal{X}), \forall\, v \in Ff\mathfrak{sp}C(\mathcal{Y})$. Hence, ψ is $Ff\mathfrak{sp}$continuous. Thus, every $Ff\mathfrak{sp}$*continuous is $Ff\mathfrak{sp}$continuous.

Note 5.2

The converse of Proposition 4.10 need not be true shows in the following

Example 5.2

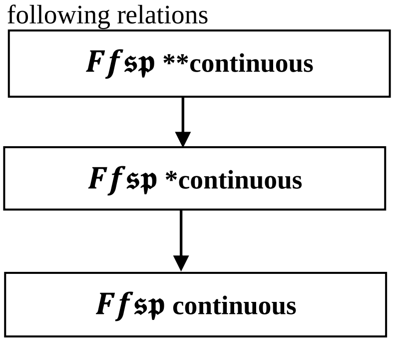
Let $\mathcal{X}$ = {a, b} be a nonempty set. Let $\zeta_1, \zeta_2 \in I^{\mathcal{X}}$ be defined by $\zeta_1(a) = 0.3$, $\zeta_1(b) = 0.4$; $\zeta_2(a) = 0.4$, $\zeta_2(b) = 0.6$. Let $\mathcal{T} = \{0,\ 1,\ \zeta_1,\ \zeta_2\}$ be the fuzzy topology on $\mathcal{X}$ and $\mathcal{T}_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{T_\alpha\}\}$ be fine fuzzy topology on $\mathcal{X}$, defined as $\mathcal{T}(\zeta_1) = \mathcal{T}_1 = \zeta_1 = \left\{ \zeta_1 \wedge \zeta_1 = \zeta_1 \neq 0_X \right\}$, $\mathcal{T}(\zeta_2) = \mathcal{T}_2 = \zeta_2 = \left\{ \zeta_2 \wedge \zeta_2 = \zeta_2 \neq 0_X \right\}$. Thus, $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$is a $\mathcal{F}f\mathcal{T}\mathcal{S}$.

Let $\mathcal{Y}$ = {c, d} be a nonempty set. Let $\delta_1, \delta_2 \in I^{\mathcal{Y}}$ be defined by $\delta_1(a) = 0.4$, $\delta_1(b) = 0.4$; $\delta_2(a) = 0.5$, $\delta_2(b) = 0.6$. Let $\mathcal{S} = \{0,\ 1,\ \delta_1,\ \delta_2\}$ be the fuzzy topology on Y and $S_f = \{0_X, 1_X, \bigcup_{\alpha\in J}\{S_\alpha\}\}$ be fine fuzzy topology on $\mathcal{Y}$, defined as $\mathcal{S}(\delta_1) = \mathcal{S}_1 = \delta_1 = \left\{ \delta_1 \wedge \delta_1 = \delta_1 \neq 0_X \right\}$, $\mathcal{S}(\delta_2) = \mathcal{S}_2 = \delta_2 = \left\{ \delta_2 \wedge \delta_2 = \delta_2 \neq 0_X \right\}$. Now $(\mathcal{Y},\mathcal{S},\mathcal{S}_f)$ be $\mathcal{F}f\mathcal{T}\mathcal{S}$.

Let $\psi: (\mathcal{X},\mathcal{T},\mathcal{T}_f) \rightarrow (\mathcal{Y},\mathcal{S},\mathcal{S}_f)$ be defined as $\psi(a) = b, \psi(b) = a$. Let $\vartheta \in I^{\mathcal{X}}$. Then '$\psi$' is $Ff\mathfrak{sp}$continuous but not $Ff\mathfrak{sp}$*continuous. $\because$, $\vartheta(a) = 0.6$, $\vartheta(b) = 0.3 \in Ff\mathfrak{sp}C(\mathcal{Y})$, but $\psi^{-1}(\vartheta) = (0.3, 0.6) \notin Ff\mathfrak{sp}C(\mathcal{X})$. Thus, ψ is not $Ff\mathfrak{sp}$*continuous. Therefore, every $Ff\mathfrak{sp}$continuous need not be $Ff\mathfrak{sp}$*continuous function.

Note 5.3

Clearly we get the following relations



(figure - 4.2)

Relations between $Ff\mathfrak{sp}$continuous & other types of Ffcontinuous map

Proposition 5.3

For a mapping $\psi: (\mathcal{X}, \mathcal{T}, \mathcal{T}_f) \rightarrow (\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$, the equivalent statements as follows.

(i) ψ is $Ff\mathfrak{sp}$continuous.

(ii) $\psi^{-1}(\lambda) \in Ff\mathfrak{sp}C(\mathcal{X}), \forall \lambda \in FfC(\mathcal{Y})$.

(iii) $FfInt(FfCl(\psi^{-1}(\lambda))) \wedge FfCl(FfInt(\psi^{-1}(\lambda))) \leq \psi^{-1}(FfCl(\lambda))$ for each fine fuzzy set λ of $\mathcal{Y}$.

(iv) $\psi\left(FfInt(FfCl(v)) \wedge \left(FfCl(FfInt(v))\right)\right) \leq FfCl(\psi(v))$ for each fine fuzzy set v of $\mathcal{X}$.

Proof

(i) $\Rightarrow$ (ii)

Let $\lambda \in FfC(\mathcal{Y})$ and $1_Y - \lambda \in FfO(\mathcal{X})$. Hence, $\psi^{-1}(1_Y - \lambda) \in Ff\mathfrak{sp}O(\mathcal{X})$. Thus, $\psi^{-1}(\lambda) \in Ff\mathfrak{sp}O(\mathcal{X})$.

(ii) $\Rightarrow$ (iii)

Assume that (ii), let fine fuzzy set λ of $\mathcal{Y}$, then $\psi^{-1}(FfCl(\lambda))$ is Ffsp closed in $\mathcal{X}$.

$$FfInt(FfCl(\psi^{-1}(\lambda))) \wedge FfCl(FfInt(\psi^{-1}(\lambda))$$

$$\leq FfInt(FfCl(\psi^{-1}(FfCl(\lambda)))) \wedge FfCl(FfInt(\psi^{-1}(FfCl(\lambda))\,))$$

$$\leq \psi^{-1}(FfCl(\lambda)).$$

(iii) $\Rightarrow$ (iv)

Let μ be fine fuzzy set of $\mathcal{X}$, put $\lambda = \psi(\mu)$ then $FfInt(FfCl(\psi^{-1}(\psi(\mu)))) \wedge FfCl(FfInt(\psi^{-1}(f(\mu))))$ $\leq \psi^{-1}(FfCl(\psi(\mu)))$ so that,

$FfInt(FfCl(\mu)) \wedge FfCl\left(FfInt((\mu))\right) \leq \psi^{-1}(FfCl(\,\psi(\mu))),$ $\psi\left(FfInt(FfCl(\mu)) \wedge FfCl\left(FfInt((\mu))\right)\right) \leq FfCl(\psi(\mu))$.

(iv) $\Rightarrow$ (i)

Let $\lambda \in FfO(\mathcal{Y})$. Put $\mu = 1_X - \lambda$ and $\mu = \psi^{-1}(\lambda)$ then

$$\psi\left(FfInt\left(FfCl(\psi^{-1}(\lambda))\right) \wedge FfCl\left(FfInt\left((\psi^{-1}(\lambda))\right)\right)\right)$$

$$\leq FfCl\left(\psi(\psi^{-1}(\lambda))\right)$$

$$\leq FfCl(\lambda) = \lambda.$$

$\psi^{-1}(\lambda) \in Ff\mathfrak{sp}C(\mathcal{X})$. Hence, ψ is $Ff\mathfrak{sp}$continuous.

Proposition 5.4

Let $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$, $(\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$ and $(\mathcal{Z}, \mathcal{R}, \mathcal{R}_f)$ be three $\mathcal{F}f\mathcal{T}\mathcal{S}s$ and $\psi: (\mathcal{X}, \mathcal{T}, \mathcal{T}_f) \rightarrow (\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$ and $\phi : (\mathcal{Y}, \mathcal{S}, \mathcal{S}_f) \rightarrow (\mathcal{Z}, \mathcal{R}, \mathcal{R}_f)$ be two maps. Then,

(i) if $\phi \circ \psi$ is $Ff\mathfrak{sp}$open and ψ is continuous surjective, then ϕ is $Ff\mathfrak{sp}O$ map.

(ii) if $\phi \circ \psi$ is FfO and ϕ is Ffcontinuous injective, then ψ is $Ff\mathfrak{sp}O$ map.

Proof

(i) Let $\eta \in FfO(\mathcal{Y})$. Then, $\psi^{-1}(\eta) \in FfO(\mathcal{X})$. $\because$ $\phi \circ \psi$ is a $Ff\mathfrak{sp}O$ map, then $(\phi \circ \psi\,)(\psi^{-1}(\eta)) = \phi\left(\psi(\psi^{-1}(\eta))\right) = \phi(\eta)$ ($\because$ ψ is surjective) is a fine fuzzy $\mathfrak{sp}$open set in $\mathcal{Z}$. Thus, ϕ is $Ff\mathfrak{sp}O$ map.

(ii) Let $\eta \in FfO(\mathcal{X})$. Then $\phi(\psi(\eta)) \in FfO(\mathcal{Z})$. $\therefore$, $\phi^{-1}(\phi(\psi(\eta))) = \psi(\eta)$ ($\because$ ϕ is injective) is a $Ff\mathfrak{sp}O(\mathcal{Y})$. Thus, ψ is $Ff\mathfrak{sp}O$ map.

Proposition 5.5

Let $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$ and $(\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$ be two $\mathcal{F}f\mathcal{TS}s$ and $\psi: (\mathcal{X}, \mathcal{T}, \mathcal{T}_f) \to (\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$ be a bijective map. Then the following are equivalent:

(i) ψ is a $Ff\mathfrak{sp}O$ map

(ii) ψ is a $Ff\mathfrak{sp}C$ map.

(iii) ψ^{-1} is a $Ff\mathfrak{sp}$ continuous map.

Proof

(i) $\Rightarrow$ (ii)

Suppose $\zeta \in FfC(\mathcal{X})$. Then $1_{\mathcal{X}} - \zeta \in FfO(\mathcal{X})$ and by (i) $(1_{\mathcal{X}} - \zeta) \in Ff\mathfrak{sp}O(\mathcal{Y})$. $\because$ ψ is bijective, then $\psi(1_{\mathcal{X}} - \zeta) = 1_{\mathcal{Y}} - \psi(\zeta)$. Hence, $\psi(\zeta)$ is $Ff\mathfrak{sp}C$ in $\mathcal{Y}$. $\therefore$, ψ is a $Ff\mathfrak{sp}C$ map.

(ii) $\Rightarrow$ (iii)

Let ψ is a $Ff\mathfrak{sp}C$ map and $\zeta \in Ff\mathfrak{sp}C(\mathcal{X})$. Since, ψ is bijective, then $(\psi^{-1})^{-1}(\zeta) = \psi(\zeta)$ which is a $Ff\mathfrak{sp}C$ set in $\mathcal{Y}$. $\therefore$, by Proposition 4.11, ψ^{-1} is $Ff\mathfrak{sp}$ continuous map.

(iii) $\Rightarrow$ (i)

Let $\delta \in FfO(\mathcal{X})$, by assumption ψ^{-1} is $Ff\mathfrak{sp}$ continuous map, then $(\psi^{-1})^{-1}(\delta) = \psi(\delta)$ which is a $Ff\mathfrak{sp}O$ set in $\mathcal{Y}$. Hence, ψ is a $Ff\mathfrak{sp}O$ map.

Proposition 5.6

Let $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$, $(\mathcal{Y}, \mathcal{S}, \mathcal{S}_f)$ and $(\mathcal{Z}, \mathcal{R}, \mathcal{R}_f)$ be three $\mathcal{F}f\mathcal{TS}s$. If $\psi: (X, T, T_f) \to (\mathcal{Y}, S, S_f)$ is $Ff\mathfrak{sp}$ continuous map and $\phi: (\mathcal{Y}, \mathcal{S}, \mathcal{S}_f) \to (\mathcal{Z}, \mathcal{R}, \mathcal{R}_f)$ is Ff continuous map then their composition $\phi \circ \psi : (\mathcal{X}, \mathcal{T}, \mathcal{T}_f) \to (\mathcal{Z}, \mathcal{R}, \mathcal{R}_f)$ is also $Ff\mathfrak{sp}$ continuous.

Proof

Let ζ be any $Ff\mathfrak{sp}$ open subset of $(\mathcal{Z}, \mathcal{R}, \mathcal{R}_f)$. Then $(\phi \circ \psi^{-1})(\zeta) = (\psi^{-1} \circ \phi^{-1})(\zeta) = \psi^{-1}(\phi^{-1}(\zeta))$. Since, ϕ is Ffcontinuous, $\phi^{-1}(\zeta)$ is fine fuzzy open in (Y, S, S_f). Since, ψ is $Ff\mathfrak{sp}$ continuous so that $\psi^{-1}(\phi^{-1}(\zeta))$ is $Ff\mathfrak{sp}$continuous in $(\mathcal{X}, \mathcal{T}, \mathcal{T}_f)$. Thus, for each $Ff\mathfrak{sp}O$set ζ in $(\mathcal{Z}, \mathcal{R}, \mathcal{R}_f)$, $(\phi \circ \psi)^{-1}(\zeta) \in Ff\mathfrak{sp}O(\mathcal{X})$. Hence, $\phi \circ \psi$ is $Ff\mathfrak{sp}$ continuous.

VI. CONCLUSION

The novel kind of space called fine fuzzy topological space is obtained by the notion fine fuzzy quasi coincident has been introduced in this article. Accordingly, the interrelations between fine fuzzy $\mathfrak{sp}$ closed sets with various types of fine fuzzy closed set have been investigated with necessary examples, which revealed that the converse need not be hold are proven with suitable examples. Then, $Ff\mathfrak{sp}$ continuous map, $Ff\mathfrak{sp}$*continuous map, $Ff\mathfrak{sp}$**continuous function and their interrelations have been established. Further, we noticed that its reverse implications are also need not be true. Finally, the fine fuzzy $\mathfrak{sp}$ open, fine fuzzy $\mathfrak{sp}$*open, fine fuzzy $\mathfrak{sp}$**open functions have been defined and its properties were briefly studied. In future, this work will be continued to investigate the fine fuzzy quotient topology, connectedness, disconnectedness and compactness in fine fuzzy bi-topological spaces and their properties.

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