On Rainbow Connection Number of Some Graphs

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Abstract: The Rainbow connection number for the following graphs, two copies of Fan graph $F_n$ by a path $P_n$, Arrow graph $A_n^2$ and $K_{1,m} \Theta K_{1,n}$, Jellyfish graph and Cycle Cactus graph have been described in this paper.

Keywords: Rainbow Coloring, Fan Graph, Arrow Graph, Corona $K_{1,m} \Theta K_{1,n}$, Jellyfish graph, Cycle Cactus graph.

I. INTRODUCTION

Finite, undirected and simple graphs are considered. An edge colored graph $G$ is rainbow edge connected, if any two vertices are connected by a path whose edges have distinct colors. Thus, the following natural parameters was defined by charted et al [1].

Let the rainbow connection of a connection graph $G$ denoted by $r_c(G)$, be the smallest number of colors, that are needed in order to make rainbow edge connected. Let $G$ be a nontrivial connected graph on which an edge coloring $C: E(G) \to \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, is defined where adjacent edges may be colored the same. Rainbow connection has an interesting application for the secure transfer of classified information between agencies, while the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. [3]

Then we consider the rainbow coloring of the following graphs,

(i) Two copies of Fan graph by a Path $P_n$
(ii) Arrow graph $A_n^2$
(iii) $K_{1,m} \Theta K_{1,n}$
(iv) Jelly Fish graph
(v) Cycle-Cactus graph
(vi) 2-Tuple graph

II. DEFINITION

A. Fan graph

A Fan graph $F_{m,n}$ is defined as the graph $\overline{K_m} + P_n$, where $\overline{K_m}$ is the empty graph on $m$ nodes and $P_n$ is the path graph on $n$ nodes, when $m = 1$, corresponds the usual Fan graph.

B. Arrow graph

An arrow graph $A_n^2$ with width 2 and length $n$ is obtained by joining a vertex $V$ with superior vertices of $p_2 \times p_n$ by 2 new edges from one end. [7]

C. Corona $K_{1,m} \Theta K_{1,n}$

$K_{1,m} \Theta K_{1,n}$ is a tree obtained by adding $n$ pendant vertices of $K_{1,m}$.

D. Jelly Fish graph

The jelly fish graph $J(m, n)$ is obtained from a 4-cycle $V_1, V_2, V_3, V_4$ by joining $V_1$ and $V_2$ with an edge and appending m pendant edges to $V_2$ and n pendant edges to $V_4$ [6]

E. Cactus

A cactus is a connected graph in which any two simple cycles have at most even vertex in common

F. 2-Tuple graph

Let $G = (V, E)$ be a simple graph and $G^3 = (V^0, E^0)$ be another copy of graph $G$. Join each vertex $v$ of $G$ to the corresponding vertex $V^0$ of $G^0$ by an edge. The new graph thus obtained we call 2-tuple graph of $G$. We denote 2-tuple graph of $G = (p, q)$ then $|V(T(G))| = 2p$ and $|T(G)| = 2p + q$.

III. MAIN RESULTS

Theorem 1:

The graph $G$ is obtained by joining two copies of fan graph $F_n$ by a path $P_k$ of length $k - 1$ [2]. Let us denote the successive vertices of first copy of Fan graph by $u_1, u_2, u_3, \ldots, u_{n+1}$ and the successive vertices of second copy of fan graph by $w_1, w_2, w_3, \ldots, w_{n+1}$. Let $v_1, v_2, v_3, \ldots, v_k$ be the path $P_k$ and $v_k = w_1$. For $n = 3$, $F_2$ is cycle $C_2$, its Rainbow coloring is discussed in [5], that the graph obtained by joining of $P_2$ and $K_2$ admits a rainbow coloring with $2n - 1$ colors.
Here we consider the case for \( n > 3 \), we define a function
\[
f : E(G) \to \{1, 2, 3, \ldots, n-1\}
\]
as follows,

- Coloring must be given,
  - \( f(u_i, v_{i+1}) = i, 1 \leq i \leq n - 2 \) \hspace{1cm} (1)
  - \( f(v_i, v_i) = 1, \text{ for all } i \) \hspace{1cm} (2)
  - \( f(w_i, v_{i+1}) = i, 1 \leq i \leq n - 2 \) \hspace{1cm} (3)
  - \( f(v_i, v_{i+2}) = i + 1, 1 \leq i \leq n - 2 \) \hspace{1cm} (4)
  - \( f(v_i, w) = n - 1, \text{ for all } i \) \hspace{1cm} (5)

**Illustrations**

Rainbow coloring of the graph obtained by joining two copies of \( P_n \) by a path \( P_2 \) shown in figure.

Thus, the rainbow coloring of the above graph is 5.

**Theorem 2:**
An Arrow graph \( A_n^2 \) which admits a Rainbow coloring, and whose Rainbow connection number is \( n \).

**Proof:**

Let \( A_n^2 \) be an Arrow graph combined by a vertex \( v_0 \) with superior vertices of \( P_2 \times P_n \) by 2 new edges. Let us denote \( v_0 \) be the starting vertex, \( v_1, v_2, \ldots, v_n \) be the upper vertices of Arrow graph, which has to be connected with \( v_0 \). Similarly, \( w_1, w_2, \ldots, w_n \) be the lower vertices of an Arrow graph, which has to be connected with \( v_0 \). Here we consider the case for \( n \geq 3 \), we define the function
\[
f : E(G) \to \{1, 2, 3, \ldots, n\}
\]
as follows, coloring has to be given Rainbow coloring of \( A_n^2 \)

- \( f(v_0, v_1) = 1 \) \hspace{1cm} (2)
- \( f(v_i, v_{i+1}) = i + 1, \text{ for } 1 \leq i \leq n \) \hspace{1cm} (3)
- \( f(w_1, v_2) = 1 \) \hspace{1cm} (4)
- \( f(v_i, w) = n - 1, \text{ for all } i \) \hspace{1cm} (5)

Thus, the rainbow coloring of the above graph is 7.

**Theorem 3:**

The graph \( K_{1,m} \Theta K_{1,n} \) for all \( m, n \) which admits Rainbow coloring, whose rainbow connection number number \( n \). Where ‘n’ represents number of edges.

**Proof:**

Let \( v_0 \) be the root of the tree. Let \( v_1, v_2, \ldots, v_m \) be the children of the root. Each subtree \( v_i, 1 \leq i \leq m \), will have ‘n’ number of vertices which have \( v_{i1}, v_{i2}, \ldots, v_{in} \) leaves [4]. The vertices that acts as a leaf of the graph \( K_{1,m} \Theta K_{1,n} \) are colored as follows.

**Illustration**
Thus the Rainbow connection number $rc(G)$ is exactly 3 as seen in (A).

It clearly shows the $rc(G)$ is the minimum number of color needed to edge coloring at least one of paths in G.

**Theorem: 5**

For every $m \geq 1, n \geq 1$ there exists a Jelly fish graph which admits a rainbow coloring with rainbow connection number is $m + n + 2$.

**Proof:**

Let G be the graph with $m + n + 4$ vertices and $m + n + 5$ edges where $m$ represents number of pendent edges in left hand side, $n$ represents its number of pendent edges in the right hand side and $E(G) = E_1 \cup E_2$ where

$E_1 = \{ux, uy, yv, vx, xy\}$

$E_2 = \{uvi, vj, 1 \leq i \leq m, 1 \leq j \leq n\}$

$ui$'s are from left pendant edges

$vi$'s are from right pendant edges

Labeling has to be defined as,

\[
\begin{align*}
    f^*: E(G) &\rightarrow \{1, 2, \ldots, m + n + 2\} \\
    f^*(xy) &= 1 \\
    f^*(ux) &= 1 \\
    f^*(yu) &= 1 \\
    f^*(yx) &= 2 \\
    f^*(vx) &= 2 \\
    f^*(uv) &= 2 \\
    f^*(ui) &= 2 + i, \text{ for } i = 1, 2, \ldots, m \\
    f^*(vj) &= 3 + m + j, \text{ for } j = 0, 1, \ldots, n - 1
\end{align*}
\]

Claim: $f^*$ is rainbow coloring

Proof: From (i) and (vii) equations found above establishes the rainbow connection number as $m + n + 2$

**Illustration:** $T^2(P_4 \times P_2)$

In the above graph $J(4,7)$ graph has Rainbow connection number as 13.

**Theorem: 6**

A cycle-cactus $C_k^{(n)}$, consist of $n$ copies of cycle $C_k$, $k \geq 3$, concatenated at exactly one vertex which holds Rainbow connection number 'k' where 'k' is the number of vertices of the cycle.

**Proof:**

Let $G_1, G_2, \ldots, G_n$ be the copies of cycles $C_k$, all concatenated at exactly one vertex namely $x_0$.

Let $x_0, x_{11}, x_{12}, x_{13} \ldots, x_{1k}$ be the vertices of $G_1$, $x_{21}, x_{22}, x_{23} \ldots, x_{2k}$ be the vertices of $G_2$, finally let $x_{n1}, x_{n2}, x_{n3} \ldots, x_{nk}$ be the vertices of $G_n$.

Let $f^*: E(G) \rightarrow \{1, 2, 3, \ldots, k\}$, then the coloring has been given as follows

\[
\begin{align*}
    f(x_0, x_{11}) &= 1 \\
    f(x_0, x_{21}) &= 1 \\
    \vdots \\
    f(x_0, x_{n1}) &= 1, \text{ for all } n \geq 1 \\
    f(x_{11}, x_{12}) &= 2 \\
    f(x_{21}, x_{22}) &= 2 \\
    \vdots \\
    f(x_{11}, x_{12}) &= 2, \text{ for all } n \geq 1 \\
    f(x_{12}, x_{13}) &= 3 \\
    f(x_{22}, x_{23}) &= 3 \\
    \vdots \\
    f(x_{12}, x_{13}) &= 3, \text{ for all } n \geq 1 \\
    f(x_{1k-1}, x_0) &= k \\
    f(x_{2k-1}, x_0) &= k \\
    \vdots \\
    f(x_{nk-1}, x_0) &= k, \text{ for all } n \geq 1
\end{align*}
\]
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**Illustration:** Cycle-Cactus $C_4^{(4)}$

In the above graph the rainbow connection number is 4.

Cycle-Cactus $C_5^{(4)}$

In the above graph the rainbow connection number is 5.

**Findings:**

<table>
<thead>
<tr>
<th>S.no</th>
<th>Graph</th>
<th>Rainbow Connection number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Two copies of Fan graph by a path</td>
<td>$n-1$ n-number of vertices</td>
</tr>
<tr>
<td>2</td>
<td>Arrow graph $A_n^{(2)}$</td>
<td>$n$ n-number of vertices</td>
</tr>
<tr>
<td>3</td>
<td>Corona graph $K_{1,m} \oplus K_{1,n}$</td>
<td>$n$ n-number of vertices</td>
</tr>
<tr>
<td>4</td>
<td>Jelly Fish graph</td>
<td>$m + n + 2$ m - Pendent edges in LHS n - Pendent edges in RHS</td>
</tr>
<tr>
<td>5</td>
<td>Cycle-Cactus graph</td>
<td>$k$ k – number of Vertices</td>
</tr>
<tr>
<td>6</td>
<td>2-Tuple graph</td>
<td>3</td>
</tr>
</tbody>
</table>

**IV. CONCLUSION:**

It is of interest to study the connection number for various classes, after than what has been found in the literature.

**REFERENCES**


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