Classifications of Pairwise Fuzzy Volterra Spaces

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Abstract: The main focus of this paper is to introduce the new types of pairwise fuzzy Volterra spaces such as by introducing pairwise fuzzy residual sets in the place of pairwise fuzzy G_δ-sets in the definition of pairwise fuzzy Volterra space, a new kind of fuzzy bitopological space namely, pairwise fuzzy ε_r-Volterra spaces has been introduced and studied and also by introducing pairwise fuzzy pre-open sets in the place of pairwise fuzzy dense sets in the definition of pairwise fuzzy Volterra space, another kind of fuzzy bitopological space namely, pairwise fuzzy ε_r-Volterra spaces has been introduced and studied. Some of their characterizations and relationships with the other fuzzy bitopological spaces have been investigated and studied.

Key words and phrases: Pairwise fuzzy dense set, pairwise fuzzy nowhere dense set, pairwise fuzzy first category set, pairwise fuzzy residual set, pairwise fuzzy Volterra space, pairwise fuzzy submaximal space, pairwise fuzzy Baire space, pairwise fuzzy nodec space and pairwise fuzzy strongly irresolvable space, .

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I. INTRODUCTION

The fuzzy sets were introduced by California University Professor Lotfi A.Zadeh in his classical paper **[16]** in 1965. The fuzzy topological spaces (FTS, in short) were introduced by C.L.Chang **[3]** in 1968. The fuzzy bitopological spaces (FBTS, in short) were introduced and studied by A.Kandil **[4]** in 1989. Recently, G.Thangaraj and S.Soundara Rajan **[12]** defined the notion of fuzzy Volterra spaces and subsequently the pairwise fuzzy Volterra spaces was introduced by the authors in **[9]**. Motivated on the generalized Volterra spaces was introduced and studied by Milan Matejdes **[5,6]** in classical topology, the concepts of generalized pairwise fuzzy Volterra spaces such as pairwise fuzzy ε_r-Volterra spaces and pairwise fuzzy ε_p-Volterra spaces have been introduced and studied in this paper and also some of their characterizations and relationships with the other FBTS have been investigated.

II. PRELIMINARIES

Some notions and results which will be needed in this paper are recalled.

Definition:2.1.[1] Let (X,T) be a FTS. For a fuzzy set α of X, the interior $int(\alpha)$ and the closure $cl(\alpha)$ are defined respectively as

(i) $int(\alpha) = \vee\{\beta : \beta \le \alpha$ and $\beta \in T\}$ and

(ii) $cl(\alpha) = \wedge\{\beta : \alpha \le \beta$ and $1-\beta \in T\}$.

Lemma:2.1.[1] Let α be any fuzzy set in a FTS (X,T). Then $1 - cl(\alpha) = int(1-\alpha)$ and $1 - int(\alpha) = cl(1-\alpha)$.

Definition:2.2.[10] A fuzzy set α in a FBTS (X,T_1,T_2) is called a pairwise fuzzy open set (PFO set, in short) if $\alpha \in T_k$, $(k = 1,2)$. The complement of PFO set is called a pairwise fuzzy closed set (PFC set, in short) in (X,T_1,T_2).

Definition:2.3.[10] A fuzzy set α in a FBTS (X,T_1,T_2) is called a pairwise fuzzy G_δ-set (PF G_δ-set, in short) if $\alpha = \wedge_{n=1}^{\infty}(\alpha_n)$, where (α_n)'s are PFO sets in (X,T_1,T_2).

Definition:2.4.[10] A fuzzy set α in a FBTS (X,T_1,T_2) is called a pairwise fuzzy F_σ-set (PF F_σ-set, in short) if $\alpha = \vee_{n=1}^{\infty}(\alpha_n)$, where (α_n)'s are PFC sets in (X,T_1,T_2).

Definition:2.5.[9] A fuzzy set α in a FBTS (X,T_1,T_2) is called a pairwise fuzzy dense set (PFD set, in short) if $cl_{T_1}cl_{T_2}(\alpha) = 1 = cl_{T_2}cl_{T_1}(\alpha)$ in (X,T_1,T_2).

Definition:2.6.[9] A fuzzy set α in a FBTS (X,T_1,T_2) is called a pairwise fuzzy nowhere dense set (PFND set, in short) if $int_{T_1}cl_{T_2}(\alpha) = 0 = int_{T_2}cl_{T_1}(\alpha)$ in (X,T_1,T_2).

Definition:2.7.[14] A fuzzy set α in a FBTS (X,T_1,T_2) is called a pairwise fuzzy first category set (PFFC set, in short) if $\alpha = \vee_{n=1}^{\infty}(\alpha_n)$, where (α_n)'s are PFND sets in (X,T_1,T_2). A fuzzy set in (X,T_1,T_2) which is not a PFFC set is called a pairwise fuzzy second category set (PFSC set, in short) in (X,T_1,T_2).

Definition:2.8.[14] If α is a PFFC set in FBTS (X,T_1,T_2), then $1-\alpha$ is called a pairwise fuzzy residual set (PFR set, in short) in (X,T_1,T_2).

Definition:2.9.[11] A fuzzy set α in a FBTS (X,T_1,T_2) is called a pairwise fuzzy σ-nowhere dense set (PF σ-nowhere dense set, in short) if α is a PF F_σ-set in (X,T_1,T_2) such that $int_{T_1}int_{T_2}(\alpha) = 0 = int_{T_2}int_{T_1}(\alpha)$.

Definition:2.10. A fuzzy set α in a FBTS (X,T_1,T_2) is called a

(i) pairwise fuzzy regular open set (PF regular open set, in short) in (X,T_1,T_2) if $int_{T_1}cl_{T_2}(\alpha) = \alpha = int_{T_2}cl_{T_1}(\alpha)$ **[2]**

(ii) pairwise fuzzy regular closed set (PF regular closed set, in short) in (X,T_1,T_2) if $cl_{T_1}int_{T_2}(\alpha) = \alpha = cl_{T_2}int_{T_1}(\alpha)$ **[2]**

(iii) pairwise fuzzy pre-open set (PF pre-open set, in short) in (X,T_1,T_2) if

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$\alpha \leq int_{T_1}cl_{T_2}(\alpha)$ and $\alpha \leq int_{T_2}cl_{T_1}(\alpha)$ **[7]**

(iv) pairwise fuzzy pre-closed set (PF pre-closed set, in short) in (X, T_1, T_2) if $cl_{T_1}int_{T_2}(\alpha) \leq \alpha$ and $cl_{T_2}int_{T_1}(\alpha) \leq \alpha$ **[7]**

(v)

Definition:2.11.[9] Let (X, T_1, T_2) be a FBTS. Then (X, T_1, T_2) is called a pairwise fuzzy Volterra space (PFVS, in short) if $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$ where (α_n)'s are PFD and PF G_δ-sets in (X, T_1, T_2).

Definition:2.12.[13] Let (X, T_1, T_2) be a FBTS. Then (X, T_1, T_2) is called a pairwise fuzzy Baire space (PF Baire space, in short) if $int_{T_k}(\vee_{n=1}^{\infty}(\alpha_n)) = 0$, $(k = 1,2)$ where (α_n)'s are PFND sets in (X, T_1, T_2).

Definition:2.13.[14] Let (X, T_1, T_2) be a FBTS. Then (X, T_1, T_2) is called a pairwise fuzzy submaximal space (PF submaximal space, in short) if every PFD set in (X, T_1, T_2) is a PFO set in (X, T_1, T_2).

Definition:2.14.[14] Let (X, T_1, T_2) be a FBTS. Then (X, T_1, T_2) is called a pairwise fuzzy nodec space (PF nodec space, in short) if every nonzero PFND set in (X, T_1, T_2) is a PFC set in (X, T_1, T_2).

Definition:2.15.[14] Let (X, T_1, T_2) be a FBTS. Then (X, T_1, T_2) is called a pairwise fuzzy strongly irresolvable space (PF strongly irresolvable space, in short) if $cl_{T_1}int_{T_2}(\alpha) = 1 = cl_{T_2}int_{T_1}(\alpha)$, for each PFD set α in (X, T_1, T_2).

1. Pairwise fuzzy ε_r-Volterra spaces

Definition:3.1. Let (X, T_1, T_2) be a FBTS. Then (X, T_1, T_2) is a pairwise fuzzy ε_r-Volterra space (PFε_rVS, in short) if $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$ where (α_n)'s are PF dense and PF residual sets in (X, T_1, T_2).

Proposition:3.1. If a FBTS (X, T_1, T_2) is a PFε_rVS, then $int_{T_k}(\vee_{n=1}^{N}(\beta_n)) = 0$, $(k = 1,2)$ where (β_n)'s are PFFC sets and $int_{T_1}int_{T_2}(\beta_n) = 0 = int_{T_2}int_{T_1}(\beta_n)$ in (X, T_1, T_2).

Proof. Let (X, T_1, T_2) be a PFε_rVS. Then $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PFD and PFR sets in (X, T_1, T_2). Now $1 - cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 0$ implies that $int_{T_k}(\vee_{n=1}^{N}(1 - \alpha_n)) = 0$. Let $\beta_n = 1 - \alpha_n$. Then $int_{T_k}(\vee_{n=1}^{N}(\beta_n)) = 0 \ldots\ldots(1)$. Since (α_n)'s are PFD sets, $cl_{T_1}cl_{T_2}(\alpha_n) = 1 = cl_{T_2}cl_{T_1}(\alpha_n)$, implies that $int_{T_1}int_{T_2}(1 - \alpha_n) = 0 = int_{T_2}int_{T_1}(1 - \alpha_n)$. Hence $int_{T_1}int_{T_2}(\beta_n) = 0 = int_{T_2}int_{T_1}(\beta_n) \ldots\ldots(2)$. Also, since (α_n)'s are PFR sets, $(1 - \alpha_n)$'s are PFFC sets in (X, T_1, T_2). That is, (β_n)'s are PFFC sets in (X, T_1, T_2). From (1) and (2), $int_{T_k}(\vee_{n=1}^{N}(\beta_n)) = 0$, where (β_n)'s are PFFC sets and $int_{T_1}int_{T_2}(\beta_n) = 0 = int_{T_2}int_{T_1}(\beta_n)$ in (X, T_1, T_2).

Proposition:3.2. If each PFND set in a PFVS (X, T_1, T_2) is a PFC set in (X, T_1, T_2), then (X, T_1, T_2) is a PFε_rVS.

Proof. Let (α_n)'s $(n = 1$ to $N)$ be PFD and PFR sets in (X, T_1, T_2). Since (α_n)'s are PFR sets, $(1 - \alpha_n)$'s are PFFC sets in (X, T_1, T_2). Then $1 - \alpha_n = \vee_{m=1}^{\infty}(\beta_{nm})$, where (β_{nm})'s are PFND sets in (X, T_1, T_2). By hypothesis, the PFND sets (β_{nm})'s are PFC sets in (X, T_1, T_2). Hence $(1 - \alpha_n)$'s are PF F_σ-sets in (X, T_1, T_2). This implies that (α_n)'s are PF G_δ-sets in (X, T_1, T_2). Then (α_n)'s are PFD and PF G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFVS, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$. Hence $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PFD and PFR sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFε_rVS.

Proposition:3.3. If a FBTS (X, T_1, T_2) is a PFVS and PF nodec space, then (X, T_1, T_2) is a PFε_rVS.

Proof. Let (α_n)'s $(n = 1$ to $N)$ be PFD and PFR sets in (X, T_1, T_2). Since (α_n)'s are PFR sets, $(1 - \alpha_n)$'s are PFFC sets in (X, T_1, T_2). Then $1 - \alpha_n = \vee_{m=1}^{\infty}(\beta_{nm})$, where (β_{nm})'s are PFND sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PF nodec space, the PFND sets (β_{nm})'s are PFC sets in (X, T_1, T_2). Therefore, by proposition 3.2, (X, T_1, T_2) is a PFε_rVS.

Theorem:3.1.[11] If α is a PFD and PF G_δ-set in a PF strongly irresolvable space (X, T_1, T_2), then α is a PFR set in (X, T_1, T_2).

Proposition:3.4. If a PFε_rVS (X, T_1, T_2) is a PF strongly irresolvable space, then (X, T_1, T_2) is a PFVS.

Proof. Let (α_n)'s $(n = 1$ to $N)$ be PFD and PF G_δ-sets in (X, T_1, T_2). Then by theorem 3.1., (α_n)'s are PFR sets in (X, T_1, T_2). This implies that (α_n)'s are PFD and PFR sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_rVS, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$. Hence, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PFD and PF G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFVS.

Theorem:3.2.[11] If α is a PFD and PFO set in a PF strongly irresolvable space (X, T_1, T_2), then $1 - \alpha$ is a PFND set in (X, T_1, T_2).

Proposition:3.5. If a FBTS (X, T_1, T_2) is a PFVS, PF strongly irresolvable space and PF submaximal space, then (X, T_1, T_2) is a PFε_rVS.

Proof. Let (α_n)'s $(n = 1$ to $N)$ be PFD and PFR sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PF submaximal space, the PFD sets (α_n)'s are PFO sets in (X, T_1, T_2). This implies that (α_n)'s are PFD and PFO sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PF strongly irresolvable space and by theorem 3.2., $(1 - \alpha_n)$'s are PFND sets in (X, T_1, T_2). Since (α_n)'s are PFO sets in (X, T_1, T_2), $(1 - \alpha_n)$'s are PFC sets in (X, T_1, T_2). Hence the PFND sets $(1 - \alpha_n)$'s are PFC sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFVS and the PFND sets $(1 - \alpha_n)$'s are PFC sets in (X, T_1, T_2) and by proposition 3.2., (X, T_1, T_2) is a PFε_rVS.

Theorem:3.3.[13] If α is a PFND set in FBTS (X, T_1, T_2), then $1 - \alpha$ is a PFD set in (X, T_1, T_2).

Proposition:3.6. If the PFFC sets in a PFε_rVS (X, T_1, T_2) are PFND sets in (X, T_1, T_2), then $int_{T_k}(\vee_{n=1}^{N}(\alpha_n)) = 0$, $(k = 1,2)$ where (α_n)'s are PFFC sets in (X, T_1, T_2).

Proof. Let (α_n)'s $(n = 1$ to $N)$ be PFFC sets in (X, T_1, T_2). Then $(1 - \alpha_n)$'s are PFR sets in (X, T_1, T_2). Since the PFFC sets (α_n)'s are PFND sets in (X, T_1, T_2) and by theorem 3.3., $(1 - \alpha_n)$'s are PFD sets in (X, T_1, T_2). This implies that $(1 - \alpha_n)$'s are PFD and PFR sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_rVS, $cl_{T_k}(\wedge_{n=1}^{N}(1 - \alpha_n)) = 1$, $(k = 1,2)$. Hence $int_{T_k}(\vee_{n=1}^{N}(\alpha_n)) = 0$, where (α_n)'s are PFFC sets in (X, T_1, T_2).

Proposition:3.7. If each PFFC set in a PF Baire space (X, T_1, T_2) is a PFND set in (X, T_1, T_2), then (X, T_1, T_2) is a PFε_rVS.

Proof. Let (α_n)'s $(n = 1 \text{ to } N)$ be PFD and PFR sets in (X, T_1, T_2). Since (α_n)'s are PFR sets, $(1-\alpha_n)$'s are PFFC sets in (X, T_1, T_2) and hence $1-\alpha_n = \vee_{m=1}^{\infty}(\beta_{nm})$, where (β_{nm})'s are PFND sets in (X, T_1, T_2). By hypothesis, $(1-\alpha_n)$'s are PFND sets in (X, T_1, T_2). Let (β_i)'s be PFND sets in (X, T_1, T_2) in which the first 'N' PFND sets be $1-\alpha_n$. Since (X, T_1, T_2) is a PF Baire space, $int_{T_k}(\vee_{i=1}^{\infty}(\beta_i)) = 0$, $(k = 1,2)$. Now $int_{T_k}(\vee_{n=1}^{N}(1-\alpha_n)) \le int_{T_k}(\vee_{i=1}^{\infty}(\beta_i))$. Then $int_{T_k}(\vee_{n=1}^{N}(1-$ *an≤0*. That is, *intTkn=1N1−an=0*. This implies that *clTkn=1Nan=1*, *(k=1,2)* where *an*'s are PFD and PFR sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFε_rVS.

Proposition:3.8. If each PFFC set in a PF strongly irresolvable space and PF Baire space (X, T_1, T_2) is a PFC set in (X, T_1, T_2), then (X, T_1, T_2) is a PFε_rVS.

Proof. Let (α_n)'s $(n = 1 \text{ to } N)$ be PFD and PFR sets in (X, T_1, T_2). Since (α_n)'s are PFR sets, $(1-\alpha_n)$'s are PFFC sets in (X, T_1, T_2). Since the PFFC sets $(1-\alpha_n)$'s are PFC sets in (X, T_1, T_2). This implies that (α_n)'s are PFO sets in (X, T_1, T_2). Then (α_n)'s are PFD and PFO sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PF strongly irresolvable space and by theorem 3.2., $(1-\alpha_n)$'s are PFND sets in (X, T_1, T_2). Let (β_i)'s be PFND sets in (X, T_1, T_2) in which the first 'N' PFND sets be $1-\alpha_n$. Since (X, T_1, T_2) is a PF Baire space, $int_{T_k}(\vee_{i=1}^{\infty}(\beta_i)) = 0$, $(k = 1,2)$. But $int_{T_k}(\vee_{n=1}^{N}(1-\alpha_n)) \le int_{T_k}(\vee_{i=1}^{\infty}(\beta_i))$ and $int_{T_k}(\vee_{i=1}^{\infty}(\beta_i)) = 0$. Then $int_{T_k}(\vee_{n=1}^{N}(1-\alpha_n)) \le 0$. That is, $int_{T_k}(\vee_{n=1}^{N}(1-\alpha_n)) = 0$. This implies that $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$ where (α_n)'s are PFD and PFR sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFε_rVS.

Theorem:3.4.[13] Let (X, T_1, T_2) be a FBTS. Then the following are equivalent:

(1) (X, T_1, T_2) is a PF Baire space.

(2) $int_{T_k}(\alpha) = 0$, $k = 1,2)$, for every PFFC set α in (X, T_1, T_2).

(3) $cl_{T_k}(\beta) = 1$, $k = 1,2)$, for every PFR set β in (X, T_1, T_2).

Proposition:3.9. If a PFε_rVS (X, T_1, T_2) is a PF Baire space, then $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$ where (α_n)'s are PFR sets in (X, T_1, T_2).

Proof. Let (α_n)'s $(n = 1 \text{ to } N)$ be PFR sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PF Baire space and by theorem 3.4., $cl_{T_k}(\alpha_n) = 1$, $(k = 1,2)$. Then $cl_{T_1}cl_{T_2}(\alpha_n) = cl_{T_1}(1) = 1$ and also $cl_{T_2}cl_{T_1}(\alpha_n) = cl_{T_2}(1) = 1$. Hence (α_n)'s are PFD and PFR sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_rVS, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$. Therefore, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PFR sets in (X, T_1, T_2).

Proposition:3.10. If (α_n)'s are PFR sets such that $\wedge_{n=1}^{N}(\alpha_n)$ is a PFR set in a PF Baire space (X, T_1, T_2), then (X, T_1, T_2) is a PFε_rVS.

Proof. Let (α_n)'s $(n = 1 \text{ to } N)$ be PFD and PFR sets in (X, T_1, T_2). Then by hypothesis, $\wedge_{n=1}^{N}(\alpha_n)$ is a PFR set in (X, T_1, T_2). Since (X, T_1, T_2) is a PF Baire space and by theorem 3.4., $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$. Therefore, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PFD and PFR sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFε_rVS.

Theorem:3.5.[10] A fuzzy set α is a PF σ-nowhere dense set in FBTS (X, T_1, T_2) if and only if $1-\alpha$ is a PFD and PF G_δ-set in (X, T_1, T_2).

Proposition:3.11. If a PFε_rVS (X, T_1, T_2) is a PF strongly irresolvable space, then $int_{T_k}(\vee_{n=1}^{N}(\alpha_n)) = 0$, where (α_n)'s are PF σ-nowhere dense sets in (X, T_1, T_2).

Proof. Let (α_n)'s $(n = 1 \text{ to } N)$ be PF σ-nowhere dense sets in (X, T_1, T_2). Then by theorem 3.5., $(1-\alpha_n)$'s are PFD and PF G_δ-sets in (X, T_1, T_2). By theorem 3.1., $(1-\alpha_n)$'s are PFR sets in (X, T_1, T_2). This implies that $(1-\alpha_n)$'s are PFD and PFR sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_rVS, $cl_{T_k}(\wedge_{n=1}^{N}(1-\alpha_n)) = 1$, $(k = 1,2)$. Therefore, $int_{T_k}(\vee_{n=1}^{N}(\alpha_n)) = 0$, where (α_n)'s are PF σ-nowhere dense sets in (X, T_1, T_2).

III. RESULTS DESCRIPTION

Definition:4.1. Let (X, T_1, T_2) be a FBTS. Then (X, T_1, T_2) is called a pairwise fuzzy ε_p-Volterra space (PFε_pVS, in short) if $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$ where (α_n)'s are PF pre-open and PF G_δ-sets in (X, T_1, T_2).

Proposition:4.1. If a FBTS (X, T_1, T_2) is a PFε_pVS, then $int_{T_k}(\vee_{n=1}^{N}(\beta_n)) = 0$, $(k = 1,2)$ where (β_n)'s are PF F_σ-sets in (X, T_1, T_2) such that $cl_{T_k}int_{T_l}(\beta_n) \le \beta_n$, $(k \ne l$ and $k, l = 1,2)$.

Proof. Let (β_n)'s $(n = 1 \text{ to } N)$ be PF F_σ-sets in (X, T_1, T_2) such that $cl_{T_k}int_{T_l}(\beta_n) \le \beta_n$, $(k \ne l$ and $k, l = 1,2)$. Then $(1-\beta_n)$'s are PF G_δ-sets in (X, T_1, T_2) and $int_{T_k}cl_{T_l}(1-\beta_n) \ge 1-\beta_n$. This implies that $(1-\beta_n)$'s are PF pre-open sets in (X, T_1, T_2). Then $(1-\beta_n)$'s are PF pre-open and PF G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_pVS, $cl_{T_k}(\wedge_{n=1}^{N}(1-$ *βn=1*. Therefore *intTkn=1Nβn=0*, *(k=1,2)* where *βn*'s are PF *Fσ*-sets in *X,T1,T2* such that *clTkintTlβn≤βn* , *(k≠l* and $k, l = 1,2)$.

Theorem:4.1.[15] If $cl_{T_1}cl_{T_2}(\alpha) = 1$ and $cl_{T_2}cl_{T_1}(\alpha) = 1$ for a fuzzy set α in a PF strongly irresolvable space (X, T_1, T_2), then $cl_{T_1}(\alpha) = 1$ and $cl_{T_2}(\alpha) = 1$ in (X, T_1, T_2).

Proposition:4.2. If a PFε_pVS (X, T_1, T_2) is a PF strongly irresolvable space, then (X, T_1, T_2) is a PFVS.

Proof. Let (α_n)'s $(n = 1 \text{ to } N)$ be PFD and PF G_δ-sets in (X, T_1, T_2). Since (α_n)'s are PFD sets in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(\alpha_n) = 1$ and $cl_{T_2}cl_{T_1}(\alpha_n) = 1$. Since (X, T_1, T_2) is a PF strongly irresolvable space and by theorem 4.1., $cl_{T_k}(\alpha_n) = 1$, $(k = 1,2)$. Now $int_{T_1}cl_{T_2}(\alpha_n) = int_{T_1}(1) = 1$ and $int_{T_2}cl_{T_1}(\alpha_n) = int_{T_2}(1) = 1$. Then $\alpha_n \le int_{T_1}cl_{T_2}(\alpha_n)$ and also $\alpha_n \le int_{T_2}cl_{T_1}(\alpha_n)$. This implies that (α_n)'s are PF pre-open sets in (X, T_1, T_2). Then (α_n)'s are PF pre-open and PF G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_pVS,

$cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$. Hence $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PFD and PF G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFVS.

Proposition:4.3. If a PFε_pVS (X, T_1, T_2) is a PF submaximal space, then (X, T_1, T_2) is a PFVS.

Proof. Let (α_n)'s $(n = 1$ to $N)$ be PFD and PF G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PF submaximal space, the PFD sets (α_n)'s are PFO sets in (X, T_1, T_2). Then $int_{T_1}(\alpha_n) = \alpha_n$ and $int_{T_2}(\alpha_n) = \alpha_n$. Now $int_{T_1}(\alpha_n) \le int_{T_1}cl_{T_2}(\alpha_n)$ and $int_{T_2}(\alpha_n) \le int_{T_2}cl_{T_1}(\alpha_n)$. Then $\alpha_n \le int_{T_1}cl_{T_2}(\alpha_n)$ and $\alpha_n \le int_{T_2}cl_{T_1}(\alpha_n)$. This implies that (α_n)'s are PF pre-open sets in (X, T_1, T_2). Then (α_n)'s are PF pre-open and PF G_δ-sets (α_n)'s in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_pVS, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$. Hence $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PFD and PF G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFVS.

Proposition:4.4. If each PF pre-closed set in a PFε_rVS and PF strongly irresolvable space (X, T_1, T_2) is a PFND set in (X, T_1, T_2), then (X, T_1, T_2) is a PFε_pVS.

Proof. Let (α_n)'s $(n = 1$ to $N)$ be PF pre-open and PF G_δ-sets in (X, T_1, T_2). Since (α_n)'s are PF pre-open sets, $(1 - \alpha_n)$'s are PF pre-closed sets in (X, T_1, T_2). By hypothesis, the PF pre-closed sets $(1 - \alpha_n)$'s are PFND sets in (X, T_1, T_2). Then by theorem 3.3., (α_n)'s are PFD sets in (X, T_1, T_2). This implies that (α_n)'s are PFD and PF G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PF strongly irresolvable space and by theorem 3.1., (α_n)'s are PFR sets in (X, T_1, T_2). Then (α_n)'s are PFD and PFR sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_rVS, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$. Hence $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PF pre-open and PF G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFε_pVS.

Proposition:4.5. If a PFε_pVS (X, T_1, T_2) is a PF strongly irresolvable space and PF submaximal space, then (X, T_1, T_2) is a PFε_rVS.

Proof. Let (α_n)'s $(n = 1$ to $N)$ be PFD and PFR sets in (X, T_1, T_2). Since (α_n)'s are PFD sets in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(\alpha_n) = 1$ and $cl_{T_2}cl_{T_1}(\alpha_n) = 1$. Since (X, T_1, T_2) is a PF strongly irresolvable space and by theorem 4.1., $cl_{T_k}(\alpha_n) = 1$, $(k = 1,2)$. Now $int_{T_1}cl_{T_2}(\alpha_n) = int_{T_1}(1) = 1$ and $int_{T_2}cl_{T_1}(\alpha_n) = int_{T_2}(1) = 1$. Then $\alpha_n \le int_{T_1}cl_{T_2}(\alpha_n)$ and $\alpha_n \le int_{T_2}cl_{T_1}(\alpha_n)$. This implies that (α_n)'s are PF pre-open sets in (X, T_1, T_2). Since (α_n)'s are PFR sets, $(1 - \alpha_n)$'s are PFFC sets in (X, T_1, T_2). Now $1 - \alpha_n = \vee_{m=1}^{\infty}(\beta_{nm})$, where (β_{nm})'s are PFND sets in (X, T_1, T_2). By theorem 3.3., $(1 - \beta_{nm})$'s are PFD sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PF submaximal space, the PFD sets $(1 - \beta_{nm})$'s are PFO sets in (X, T_1, T_2). Now $\alpha_n = 1 - (1 - \alpha_n) = 1 - \vee_{m=1}^{\infty}(\beta_{nm}) = \wedge_{m=1}^{\infty}(1 - \beta_{nm})$. Since $(1 - \beta_{nm})$'s are PFO sets, (α_n)'s are PF G_δ-sets in (X, T_1, T_2). This implies that (α_n)'s are PF pre-open and PF G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_pVS, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$. Hence $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PFD and PFR sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFε_rVS.

Proposition:4.6. If each PFND set in a PFε_pVS and PF strongly irresolvable space (X, T_1, T_2) is a PFC set in (X, T_1, T_2), then (X, T_1, T_2) is a PFε_rVS.

Proof. Let (α_n)'s $(n = 1$ to $N)$ be PFD and PFR sets in (X, T_1, T_2). Since (α_n)'s are PFD sets in (X, T_1, T_2), $cl_{T_1}cl_{T_2}(\alpha_n) = 1$ and $cl_{T_2}cl_{T_1}(\alpha_n) = 1$. Since (X, T_1, T_2) is a PF strongly irresolvable space and by theorem 4.1., $cl_{T_k}(\alpha_n) = 1$, $(k = 1,2)$. Now $int_{T_1}cl_{T_2}(\alpha_n) = int_{T_1}(1) = 1$ and $int_{T_2}cl_{T_1}(\alpha_n) = int_{T_2}(1) = 1$. Then $\alpha_n \le int_{T_1}cl_{T_2}(\alpha_n)$ and $\alpha_n \le int_{T_2}cl_{T_1}(\alpha_n)$. This implies that (α_n)'s are PF pre-open sets in (X, T_1, T_2). Since (α_n)'s are PFR sets, $(1 - \alpha_n)$'s are PFFC sets in (X, T_1, T_2). Therefore $1 - \alpha_n = \vee_{m=1}^{\infty}(\beta_{nm})$, where (β_{nm})'s are PFND sets in (X, T_1, T_2). By hypothesis, the PFND sets (β_{nm})'s are PFC sets in (X, T_1, T_2). Then $(1 - \beta_{nm})$'s are PFO sets in (X, T_1, T_2). Now $\alpha_n = 1 - (1 - \alpha_n) = 1 - \vee_{m=1}^{\infty}(\beta_{nm}) = \wedge_{m=1}^{\infty}(1 - \beta_{nm})$. Since $(1 - \beta_{nm})$'s are PFO sets, (α_n)'s are PF G_δ-sets in (X, T_1, T_2). Then (α_n)'s are PF pre-open and PF G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a PFε_pVS, $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, $(k = 1,2)$. Hence $cl_{T_k}(\wedge_{n=1}^{N}(\alpha_n)) = 1$, where (α_n)'s are PFD and PFR sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFε_rVS.

Proposition:4.7. If a PFε_pVS (X, T_1, T_2) is a PF nodec space and PF strongly irresolvable space, then (X, T_1, T_2) is a PFε_rVS.

Proof. Let (X, T_1, T_2) be a PFε_pVS, PF nodec space and PF strongly irresolvable space. Since (X, T_1, T_2) is a PF nodec space, each PFND set is a PFC set in (X, T_1, T_2). Therefore, by proposition 4.6., (X, T_1, T_2) is a PFε_rVS.

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