Classifications of Pairwise Fuzzy Volterra Spaces

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Abstract: The main focus of this paper is to introduce the new types of pairwise fuzzy Volterra spaces such as by introducing pairwise fuzzy residual sets in the place of pairwise fuzzy Gε-sets in the definition of pairwise fuzzy Volterra space, a new kind of fuzzy bitopological space namely, pairwise fuzzy εε-Volterra spaces has been introduced and studied and also by introducing pairwise fuzzy pre-open sets in the place of pairwise fuzzy dense sets in the definition of pairwise fuzzy Volterra space, another kind of fuzzy bitopological space namely, pairwise fuzzy εε-Volterra spaces has been introduced and studied. Some of their characterizations and relationships with the other fuzzy bitopological spaces have been investigated and studied.

Key words and phrases: Pairwise fuzzy dense set, pairwise fuzzy nowhere dense set, pairwise fuzzy first category set, pairwise fuzzy residual set, pairwise fuzzy Volterra space, pairwise fuzzy submaximal space, pairwise fuzzy Baire space, pairwise fuzzy nodec space and pairwise fuzzy strongly irresolvable space.

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I. INTRODUCTION

The fuzzy sets were introduced by California University Professor Lotfi A.Zadeh in his classical paper [16] in 1965. The fuzzy topological spaces (FTS, in short) were introduced by C.L.Chang [3] in 1968. The fuzzy bitopological spaces (FBTS, in short) were introduced and studied by A.Kandil [4] in 1989. Recently, G.Thangaraj and S.Soundara Rajan [12] defined the notion of fuzzy Volterra spaces and subsequently the pairwise fuzzy Volterra spaces was introduced by the authors in [9]. Motivated on the generalized Volterra spaces was introduced and studied by Milan Matejdes [5,6] in classical topology, the concepts of generalized pairwise fuzzy Volterra spaces such as pairwise fuzzy εε-Volterra spaces and pairwise fuzzy εε-Volterra spaces have been introduced and studied in this paper and also some of their characterizations and relationships with the other FBTS have been investigated.

II. PRELIMINARIES

Some notions and results which will be needed in this paper are recalled.

Definition 2.1.1 Let (X, T) be a FTS. For a fuzzy set α of X, the interior int (α) and the closure cl (α) are defined respectively as
(i) \( \text{int}(\alpha) = \{\beta \in T : \beta \subseteq \alpha \} \)
(ii) \( \text{cl}(\alpha) = \{\beta \in T : \beta \supseteq \alpha \} \).

Lemma 2.1.1 Let α be any fuzzy set in a FTS (X, T). Then \( 1 - cl(\alpha) = int(1 - \alpha) \) and \( 1 - \text{int}(\alpha) = cl(1 - \alpha) \).

Definition 2.2.10 A fuzzy set α in a FBTS (X, T1, T2) is called a pairwise fuzzy open set (PFO set, in short) if \( \alpha \in T_k \) for some \( k = 1,2 \). The complement of PFO set is called a pairwise fuzzy closed set (PFSC set, in short) in \( (X, \{T_1, T_2\}) \).

Definition 2.3.10 A fuzzy set α in a FBTS (X, T1, T2) is called a pairwise fuzzy Gδ-set (PF Gδ-set, in short) if \( \alpha = \bigcap_{n=1}^{\infty} \alpha_n \) where \( \{\alpha_n\} \) are PFO sets in \( (X, T_1, T_2) \).

Definition 2.4.10 A fuzzy set α in a FBTS (X, T1, T2) is called a pairwise fuzzy Fσ-set (PF Fσ-set, in short) if \( \alpha = \bigcup_{n=1}^{\infty} \alpha_n \) where \( \{\alpha_n\} \) are PFSC sets in \( (X, T_1, T_2) \).

Definition 2.5.9 A fuzzy set α in a FBTS (X, T1, T2) is called a pairwise fuzzy dense set (PFDS set, in short) if \( cl_1cl_2(\alpha) = cl_1cl_2(\alpha) = 1 = cl_1cl_2(\alpha) \) in \( (X, T_1, T_2) \).

Definition 2.6.9 A fuzzy set α in a FBTS (X, T1, T2) is called a pairwise fuzzy nowhere dense set (PFND set, in short) if \( int_1cl_2(\alpha) = 0 = int_2cl_1(\alpha) \) in \( (X, T_1, T_2) \).

Definition 2.7.14 A fuzzy set α in a FBTS (X, T1, T2) is called a pairwise fuzzy first category set (PF FFC set, in short) if \( \alpha = \bigcup_{n=1}^{\infty} \alpha_n \) where \( \{\alpha_n\} \) are PFND sets in \( (X, T_1, T_2) \). A fuzzy set in \( (X, T_1, T_2) \) which is not a PF FFC set is called a pairwise fuzzy second category set (PF FSC set, in short) in \( (X, T_1, T_2) \).

Definition 2.8.14 If α is a PF FFC set in FBTS \( (X, T_1, T_2) \), then \( 1 - \alpha \) is called a pairwise fuzzy residual set (PF R set, in short) in \( (X, T_1, T_2) \).

Definition 2.9.11 A fuzzy set α in a FBTS \( (X, T_1, T_2) \) is called a pairwise fuzzy σ-nowhere dense set (PF σ-nowhere dense set, in short) if α is a PF Fσ-set in \( (X, T_1, T_2) \) such that \( int_1int_2(\alpha) = 0 = int_2int_1(\alpha) \).

Definition 2.10. A fuzzy set α in a FBTS \( (X, T_1, T_2) \) is called a
(i) pairwise fuzzy regular open set (PF regular open set, in short) in \( (X, T_1, T_2) \) if \( int_1cl_2(\alpha) = int_2cl_1(\alpha) \)
(ii) pairwise fuzzy regular closed set (PF regular closed set, in short) in \( (X, T_1, T_2) \) if \( cl_1int_2(\alpha) = cl_2int_1(\alpha) \)
(iii) pairwise fuzzy pre-open set (PF pre-open set, in short) in \( (X, T_1, T_2) \) if...
\[ \alpha \leq \text{int}_{r_1} c_{r_2} (\alpha) \quad \text{and} \quad \alpha \leq \text{int}_{r_2} c_{r_1} (\alpha) \quad [7] \]

(iv) \( \text{pairwise fuzzy pre-closed set (PF pre-closed set, in short): } (X, T_{1}, T_{2}) \) if \( c_{r_1} \text{int}_{r_2} (\alpha) \leq \alpha \) and \( c_{r_2} \text{int}_{r_1} (\alpha) \leq \alpha \) \[7\]

(v)

**Definition 2.11 [9]** Let \((X, T_{1}, T_{2})\) be a BFTS. Then \((X, T_{1}, T_{2})\) is called a pairwise fuzzy Volterra space (PFVS, in short) if \(c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 1, (k = 1, 2)\) where \(\alpha_{n}\)'s are PF and PF \(G_{s}\)-sets in \((X, T_{1}, T_{2})\).

**Definition 2.12 [13]** Let \((X, T_{1}, T_{2})\) be a BFTS. Then \((X, T_{1}, T_{2})\) is called a pairwise fuzzy Baire space (PF Baire space, in short) if \(\text{int}_{r_1} (V_{n=1}^{n}(\alpha_{n})) = 0, (k = 1, 2)\) where \(\alpha_{n}\)'s are PFND sets in \((X, T_{1}, T_{2})\).

**Definition 2.13 [14]** Let \((X, T_{1}, T_{2})\) be a BFTS. Then \((X, T_{1}, T_{2})\) is called a pairwise fuzzy submaximal space (PF submaximal space, in short) if every PF set in \((X, T_{1}, T_{2})\) is a PFO set in \((X, T_{1}, T_{2})\).

**Definition 2.14 [14]** Let \((X, T_{1}, T_{2})\) be a BFTS. Then \((X, T_{1}, T_{2})\) is called a pairwise fuzzy nodec space (PF nodec space, in short) if every nonzero PFND set in \((X, T_{1}, T_{2})\) is a PFC set in \((X, T_{1}, T_{2})\).

**Definition 2.15 [14]** Let \((X, T_{1}, T_{2})\) be a BFTS. Then \((X, T_{1}, T_{2})\) is called a pairwise fuzzy strongly irresolvable space (PF strongly irresolvable space, in short) if \(c_{r_1} \text{int}_{r_2} (\alpha) = 1 = c_{r_2} \text{int}_{r_1} (\alpha)\), for each PF set \(\alpha\) in \((X, T_{1}, T_{2})\).

1. **Pairwise fuzzy \(e_{\gamma}\)-Volterra spaces**

**Definition 3.1.** Let \((X, T_{1}, T_{2})\) be a BFTS. Then \((X, T_{1}, T_{2})\) is a pairwise fuzzy \(e_{\gamma}\)-Volterra space (PF\(e_{\gamma}\)-VS, in short) if \(c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 1, (k = 1, 2)\) where \(\alpha_{n}\)'s are PF dense and PF residual sets in \((X, T_{1}, T_{2})\).

**Proposition 3.1.** If a BFTS \((X, T_{1}, T_{2})\) is a PF\(e_{\gamma}\)-VS, then \(c_{r_1} (\Lambda_{n=1}^{n}(\beta_{n})) = 0, (k = 1, 2)\) where \(\beta_{n}\)'s are PFFC sets and \(c_{r_2} \text{int}_{r_1} (\beta_{n}) = 0 = c_{r_2} \text{int}_{r_1} (\beta_{n})\) in \((X, T_{1}, T_{2})\).

**Proof.** Let \((X, T_{1}, T_{2})\) be a PF\(e_{\gamma}\)-VS. Then \(c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 1\), where \(\alpha_{n}\)'s are PF and PFR sets in \((X, T_{1}, T_{2})\). Now \(1 = c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 0 = c_{r_2} (\text{int}_{r_1} (1 - \alpha_{n})) = 0\). Let \(\beta_{n} = 1 - \alpha_{n}\). Then \(c_{r_2} (\text{int}_{r_1} (\beta_{n})) = 0 = c_{r_2} (\text{int}_{r_1} (\beta_{n}))\). Since \(\alpha_{n}\)'s are PF sets, \(c_{r_1} (\text{int}_{r_2} (\beta_{n})) = 1 = c_{r_2} (\text{int}_{r_1} (\alpha_{n})\). Implying that \(c_{r_1} (\text{int}_{r_2} (1 - \alpha_{n})) = 0 = c_{r_2} (\text{int}_{r_1} (1 - \alpha_{n}))\). Hence \(c_{r_1} (\text{int}_{r_2} (\beta_{n})) = 0 = c_{r_2} (\text{int}_{r_1} (\beta_{n}))\). Also, since \(\alpha_{n}\)'s are PFR sets, \(c_{r_2} (\text{int}_{r_1} (1 - \alpha_{n})) = 0 = c_{r_2} (\text{int}_{r_1} (1 - \alpha_{n}))\). Hence \(c_{r_1} (\text{int}_{r_2} (\beta_{n})) = 0 = c_{r_2} (\text{int}_{r_1} (\beta_{n}))\). Therefore \((X, T_{1}, T_{2})\) is a PF\(e_{\gamma}\)-VS.

**Proposition 3.2.** If each PFND set in a PFVS \((X, T_{1}, T_{2})\) is a PFC set in \((X, T_{1}, T_{2})\), then \((X, T_{1}, T_{2})\) is a PF\(e_{\gamma}\)-VS.

**Proof.** Let \(\alpha_{n}\)'s \((n = 1 \text{ to } N)\) be PF\(D\) and PFR sets in \((X, T_{1}, T_{2})\). Then \(1 - \alpha_{n} = \text{int}_{r_1} (\beta_{n})\), where \(\beta_{n}\)'s are PFND sets in \((X, T_{1}, T_{2})\). By hypothesis, the PFND sets \(\beta_{n}\)'s are PFC sets in \((X, T_{1}, T_{2})\). From \(1 - \alpha_{n}\)'s are PFR sets in \((X, T_{1}, T_{2})\). This implies that \(\alpha_{n}\)'s are PF \(G_{s}\)-sets in \((X, T_{1}, T_{2})\). Then \(\alpha_{n}\)'s are PF\(D\) and PF \(G_{s}\)-sets in \((X, T_{1}, T_{2})\). Since \((X, T_{1}, T_{2})\) is a PFVS, \(c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 1, (k = 1, 2)\). Hence \(c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 1, (k = 1, 2)\). Therefore \((X, T_{1}, T_{2})\) is a PF\(e_{\gamma}\)-VS.

**Theorem 3.1.1.** If \(\alpha\) is a PF and a \(PF_{G}\)-set in a strongly irresolvable space \((X, T_{1}, T_{2})\), then \(\alpha\) is a PFR set in \((X, T_{1}, T_{2})\).

**Proposition 3.4.** If a PF\(e_{\gamma}\)-VS \((X, T_{1}, T_{2})\) is a PF strongly irresolvable space, then \((X, T_{1}, T_{2})\) is a PFVS.

**Proof.** Let \(\alpha_{n}\)'s \((n = 1 \text{ to } N)\) be PF\(D\) and PF \(G_{s}\)-sets in \((X, T_{1}, T_{2})\). Then \(c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 1, (k = 1, 2)\). Hence \(c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 1, (k = 1, 2)\). Therefore \((X, T_{1}, T_{2})\) is a PFVS.

**Theorem 3.2.1.** If \(\alpha\) is a PF and a PFO set in a strongly irresolvable space \((X, T_{1}, T_{2})\), then \(1 - \alpha\) is a PFND set in \((X, T_{1}, T_{2})\).

**Proposition 3.5.** If a BFTS \((X, T_{1}, T_{2})\) is a PFVS, PF strongly irresolvable space and PF submaximal space, then \((X, T_{1}, T_{2})\) is a PF\(e_{\gamma}\)-VS.

**Proof.** Let \(\alpha_{n}\)'s \((n = 1 \text{ to } N)\) be PF\(D\) and PFR sets in \((X, T_{1}, T_{2})\). Then \(c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 1, (k = 1, 2)\). Hence \(c_{r_1} (\Lambda_{n=1}^{n}(\alpha_{n})) = 1, (k = 1, 2)\). Therefore \((X, T_{1}, T_{2})\) is a PF\(e_{\gamma}\)-VS.
Proposition 3.7. If each PFFC set in a PF Baire space $(X,T_1,T_2)$ is a PFFC set in $(X,T_1,T_2)$, then $(X,T_1,T_2)$ is a PFFC VS. 
Proof. Let $(a_n)$’s $(n=1$ to $N)$ be PFFC and PFR sets in $(X,T_1,T_2)$. Since $(a_n)$’s are PFR sets, $(1 - a_n)$’s are PFFC sets in $(X,T_1,T_2)$ and hence $1 - a_n = V_n^a = V_n^{a_n}$. Where $(b_n)$’s are PFR sets in $(X,T_1,T_2)$. By hypothesis, $(1 - a_n)$’s are PFR sets in $(X,T_1,T_2)$. Let $(b_n)$’s be PFR sets in $(X,T_1,T_2)$ in which the first ’N’ PFFC sets be $1 - a_n$. Since $(X,T_1,T_2)$ is a PF Baire space, $\int_{\tau_k} (V_n^{a_n}(\beta_n)) = 0, (k=1,2).$ Now $\int_{\tau_k} (V_n^{a_n}(1 - a_n)) \leq \int_{\tau_k} (V_n^a(\beta_n))$. Then $\int_{\tau_k} (V_n^{a_n}(1 - a_n)) \leq 0$. That is, $\int_{\tau_k} n = 1/\alpha = 0$. This implies that $\tau_k n = 1/\alpha = 1, (k=1,2)$ where $\alpha$’s are PFD and PFR sets in $(X,T_1,T_2)$. Therefore $(X,T_1,T_2)$ is a PFFC VS. 

Proposition 3.8. If each PFFC set in a PF strongly irresolvable space and PF Baire space $(X,T_1,T_2)$ is a PFFC set in $(X,T_1,T_2)$, then $(X,T_1,T_2)$ is a PFFC VS. 
Proof. Let $(a_n)$’s $(n=1$ to $N)$ be PFFC and PFR sets in $(X,T_1,T_2)$. Since $(a_n)$’s are PFR sets, $(1 - a_n)$’s are PFFC sets in $(X,T_1,T_2)$ and hence $1 - a_n = V_n^a = V_n^{a_n}$. Where $(b_n)$’s are PFR sets in $(X,T_1,T_2)$. Then $(1 - a_n)$’s are PFD and PFR sets in $(X,T_1,T_2)$. Since $(X,T_1,T_2)$ is a PF strongly irresolvable space and by theorem 3.2, $(1 - a_n)$’s are PFFC and PFR sets in $(X,T_1,T_2)$. Let $(b_n)$’s be PFR sets in $(X,T_1,T_2)$ in which the first ’N’ PFFC sets be $1 - a_n$. Since $(X,T_1,T_2)$ is a PF Baire space, $\int_{\tau_k} (V_n^a(\beta_n)) = 0, (k=1,2).$ But $\int_{\tau_k} (V_n^{a_n}(1 - a_n)) \leq \int_{\tau_k} (V_n^{a_n}(\beta_n))$ and $\int_{\tau_k} (V_n^{a_n}(\beta_n)) = 0$. Then $\int_{\tau_k} (V_n^{a_n}(1 - a_n)) \leq 0$. That is, $\int_{\tau_k} (V_n^{a_n}(1 - a_n)) = 0$. This implies that $\tau_k (V_n^{a_n}(1 - a_n)) = 1, (k=1,2)$ where $(a_n)$’s are PFD and PFR sets in $(X,T_1,T_2)$. Therefore $(X,T_1,T_2)$ is a PFFC VS. 

Theorem 3.4 [13]. Let $(X,T_1,T_2)$ be a FBTS. Then the following are equivalent: 
1. $(X,T_1,T_2)$ is a PF Baire space. 
2. $(\tau_k, (\psi_k))$, for every PFFC set $(\alpha, (k=1,2))$. 
3. $(\tau_k, (\psi_k))$, for every PFFC set $(\beta, (k=1,2))$. 

Proposition 3.9. If a PFFC VS $(X,T_1,T_2)$ is a PF Baire space, then $\tau_k (\Lambda_n^{a_n}(a_n)) = 1, (k=1,2)$ where $(\alpha)$’s are PFR sets in $(X,T_1,T_2)$. 
Proof. Let $(a_n)$’s $(n=1$ to $N)$ be PFR sets in $(X,T_1,T_2)$. Since $(X,T_1,T_2)$ is a PF Baire space and by theorem 3.4, $\tau_k (\alpha) = 1, (k=1,2)$. Then $\tau_k (\beta) = 1$ and also $\tau_k (\beta) = 1$. Hence $(\alpha)$’s are PFD and PFR sets in $(X,T_1,T_2)$. Since $(X,T_1,T_2)$ is a PF Baire space and by theorem 3.4, $\tau_k (\Lambda_n^{a_n}(a_n)) = 1, (k=1,2)$. Therefore, $\tau_k (\Lambda_n^{a_n}(a_n)) = 1$ where $(\alpha)$’s are PFR sets in $(X,T_1,T_2)$. 

Proposition 3.10. If $(\alpha_n)$’s are PFR sets such that $\Lambda_n^{a_n}(a_n)$ is a PFR set in a PF Baire space $(X,T_1,T_2)$, then $(X,T_1,T_2)$ is a PFFC VS. 
Proof. Let $(\alpha_n)$’s $(n=1$ to $N)$ be PFD and PFR sets in $(X,T_1,T_2)$. Then by hypothesis, $\Lambda_n^{a_n}(a_n)$ is a PFR set in $(X,T_1,T_2)$. Since $(X,T_1,T_2)$ is a PF Baire space and by theorem 3.4, $\tau_k (\Lambda_n^{a_n}(a_n)) = 1, (k=1,2)$. Therefore, $\tau_k (\Lambda_n^{a_n}(a_n)) = 1$. Since $(\alpha_n)$’s are PFD and PFR sets in $(X,T_1,T_2)$. Therefore $(X,T_1,T_2)$ is a PFFC VS. 

Theorem 3.5. A fuzzy set $\alpha$ is a PF $\sigma$-nowhere dense set in FBTS $(X,T_1,T_2)$ if only if $1 - \alpha$ is a PFD and PF $G_\delta$-set in $(X,T_1,T_2)$. 

Proposition 3.11. If a PFFC VS $(X,T_1,T_2)$ is a PF strongly irresolvable space, then $\tau_k (V_n^{a_n}(\alpha)) = 0$, where $(\alpha)$’s are PF $\sigma$-nowhere dense sets in $(X,T_1,T_2)$. 

Proof. Let $(\alpha_n)$’s $(n=1$ to $N)$ be PF $\sigma$-nowhere dense sets in $(X,T_1,T_2)$. Then by theorem 3.5, $(1 - \alpha_n)$’s are PFD and PF $G_\delta$-sets in $(X,T_1,T_2)$. By theorem 3.1, $(1 - \alpha_n)$’s are PFR sets in $(X,T_1,T_2)$. This implies that $(1 - \alpha_n)$’s are PFD and PFR sets in $(X,T_1,T_2)$. Since $(X,T_1,T_2)$ is a PF Baire space, $\tau_k (\Lambda_n^{a_n}(1 - \alpha_n)) = 1$. Then $\tau_k (\Lambda_n^{a_n}(1 - \alpha_n)) = 0$ where $(\alpha_n)$’s are PF $\sigma$-nowhere dense sets in $(X,T_1,T_2)$. 

III. RESULTS DESCRIPTION 

Definition 4.1. Let $(X,T_1,T_2)$ be a NBTS. Then $(X,T_1,T_2)$ is called a pairwise fuzzy $\varepsilon_p$-Volterra space (PFFC VS in short) if $\tau_k (\Lambda_n^{a_n}(a_n)) = 1, (k=1,2)$ where $(\alpha_n)$’s are PF pre-open and PF $G_\delta$-sets in $(X,T_1,T_2)$. 

Proposition 4.1. If a NBTS $(X,T_1,T_2)$ is a PFFC VS, then $\tau_k (V_n^{a_n}(\alpha)) = 0, (k=1,2)$ where $(\beta)$’s are PF F$_\delta$-sets in $(X,T_1,T_2)$ such that $\tau_k (\int_{\tau_k} (V_n^{a_n}(\beta_n)) \leq \beta_n, (k \neq l \text{ and } k,l = 1,2)$. 

Proof. Let $(\beta_n)$’s $(n=1$ to $N)$ be PF F$_\delta$-sets in $(X,T_1,T_2)$ such that $\tau_k (\int_{\tau_k} (V_n^{a_n}(\beta_n)) \leq \beta_n, (k \neq l \text{ and } k,l = 1,2).$ Then $(1 - \beta_n)$’s are PF $G_\delta$-sets in $(X,T_1,T_2)$ and $\tau_k (\int_{\tau_k} (1 - \beta_n) \geq 1 - \beta_n$. This implies that $(1 - \beta_n)$’s are PFR sets in $(X,T_1,T_2)$. Then $(1 - \beta_n)$’s are PF pre-open and PF $G_\delta$-sets in $(X,T_1,T_2)$. Since $(1 - \beta_n)$’s are PFR sets in $(X,T_1,T_2)$ is a PFFC VS, $\tau_k (\Lambda_n^{a_n}(1 - \beta_n)) = 1$. Therefore $\int_{\tau_k} (1 - \beta_n) = 0, (k=1,2)$ where $\beta_n$’s are PF $\sigma$-nowhere dense sets in $(X,T_1,T_2)$ such that $\tau_k (\int_{\tau_k} (V_n^{a_n}(\beta_n)) \leq \beta_n, (k \neq l$ and $k,l = 1,2)$. 

Theorem 4.1. Let $(X,T_1,T_2)$ be a NFBS. Then $(X,T_1,T_2)$ is called a pairwise fuzzy $\varepsilon_p$-Volterra space (PFFC VS in short) if $\tau_k (\Lambda_n^{a_n}(a_n)) = 1, (k=1,2)$ where $(\alpha_n)$’s are PF pre-open and PF $G_\delta$-sets in $(X,T_1,T_2)$. 

Proof. Let $(\beta_n)$’s $(n=1$ to $N)$ be PFD and PF $G_\delta$-sets in $(X,T_1,T_2)$. Since $(\alpha_n)$’s are PFD sets in $(X,T_1,T_2), \tau_k (\Lambda_n^{a_n}(a_n)) = 1, (k=1,2)$ when $(\alpha_n)$’s are PFD and PF $G_\delta$-sets in $(X,T_1,T_2)$.
cl_{T_k}(A_{n=1}^N(a_n)) = 1, (k = 1, 2). Hence cl_{T_k}(A_{n=1}^N(a_n)) = 1, where (a_n)'s are PFD and PF G_k-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a PFVS.

**Proposition 4.3.** If a PFVS (X, T_1, T_2) is a submaximal space, then (X, T_1, T_2) is a PFVS.

**Proof.** Let (a_n)'s (n = 1 to N) be PFD and PF G_k-sets in (X, T_1, T_2). Since (a_n)'s are PF pre-open sets, (1 - a_n)'s are PF strongly irresolvable space and by theorem 4.1., cl_{T_k}(a_n) = 1, (k = 1, 2).

**REFERENCES**