

Some Results on Prime and Distinct Prime Distance Labeling of Graphs

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Abstract: Let Z be the set of all integers. A graph H is a prime distance graph if there exists an injective function $g: V(H) \rightarrow Z$ such that for any two adjacent vertices x and y , the integer $|g(x) - g(y)|$ is a prime. So H is a prime distance graph if and only if there exists a prime distance labeling of H . If the edge labels of H are also distinct, then g is called a distinct prime distance labeling of H and H is called a distinct prime distance graph. The generalized Petersen graphs $P(n, k)$ are defined to be a graph on $2n(n \geq 3)$ vertices with $V(P(n, k)) = \{v_i, u_i: 0 \leq i \leq n - 1\}$ and $E(P(n, k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k}: 0 \leq i \leq n - 1, \text{subscripts modulo } n\}$. In this paper, we show that the generalized Petersen graphs $P(n, 3)$ permit a prime distance labeling for all even $n > 5$ and conjecture that $P(n, 2)$ and $P(n, 3)$ admit a prime distance labeling for any $n \geq 5$ and all odd $n \geq 5$, respectively. We also prove that the cycle C_n admits a distinct prime distance labeling for all $n \geq 3$, besides establishing the prime distance labeling for some graphs.

Keywords: Prime Distance Graphs, Prime Distance Labeling, Distinct Prime Distance Labeling, Generalized Petersen Graphs.

I. INTRODUCTION

Only simple, finite, connected, and undirected graphs are considered throughout this paper. The concept of distance graph was introduced by Eggleton et al. [1]. Let D be a subset of the set of all positive integers. The integer distance graph $G(Z, D)$ is the graph with vertex set Z (the set of all integers) and two vertices s and t are adjacent if and only if $|s - t| \in D$. Then the prime distance graph $G(Z, P)$ is the distance graph with $D = P$, the set of all primes. For a detailed study on integer distance and prime distance graphs, see [13-15].

Laison et al. [4] considered the finite subgraphs of $G(Z, P)$. They defined that a graph H is a prime distance graph if there exists an injective function $g: V(H) \rightarrow Z$ such that for any two vertices s and t which are adjacent, the integer $|g(s) - g(t)|$ is a prime number and g is called a prime distance labeling of H . So H is a prime distance graph if and only if there exists a prime distance labeling of H . Note that in a prime distance labeling, the labels on the vertices of H must be distinct, but the edge labels need not be so. For more results on prime distance labeling one can see [6-10].

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The generalized Petersen graphs $P(n, k)$ are defined to be a graph on $2n(n \geq 3)$ vertices with $V(P(n, k)) = \{v_i, u_i: 0 \leq i \leq n - 1\}$ and $E(P(n, k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k}: 0 \leq i \leq n - 1, \text{subscripts modulo } n\}$ [5]. In this paper we prove that the generalized Petersen graphs $P(n, 3)$ permit a prime distance labeling for all even $n > 5$ and conjecture that $P(n, 2)$ and $P(n, 3)$ admit a prime distance labeling for any $n \geq 5$ and all odd $n \geq 5$, respectively. We also establish the prime distance labeling of certain graphs. For the sake of convenience, by GPG, we mean the generalized Petersen graph.

II. MAIN RESULTS

First, we recall a few important results concerning prime distance labeling of graphs.

Lemma 1 [4] Every subgraph of a prime distance graph is also a prime distance graph.

Theorem 1 [4] All bipartite graphs permit a prime distance labelling.

Theorem 2 [7] The wheel graph $W_n = C_{n-1} \wedge K_1$ does not admit a prime distance labeling for $n \geq 10$.

Theorem 3 [7] The fan graph $F_{(1,n)}$ does not admit a prime distance labeling for $n \geq 11$.

A. Prime Distance Labelling of GPG $P(n, 2)$ and $P(n, 3)$

Lemma 2 The GPG $P(n, 2)$ is not bipartite for any $n \geq 5$.

Lemma 3 The GPG $P(n, 3)$, $n \geq 5$ is bipartite when n is even and not bipartite when n is odd.

Theorem 4 The GPG $P(n, 3)$ permits a prime distance labeling for all even $n > 5$.

Proof. The proof is direct from Lemma 3 and Theorem 1.

We propose the following two conjectures which could be proved once one can provide a pattern of prime distance labeling of them. We give the prime distance labeling of $P(7, 3)$ and $P(8, 2)$ in support of these conjectures (See Figure 1).

Conjecture 1 The GPG $P(n, 2)$ permits a prime distance labeling for any $n \geq 5$.

Conjecture 2 The GPG $P(n, 3)$ permits a prime distance labeling for all odd $n \geq 5$.

B. Prime Distance Labelling of Certain Graphs

Definition 5 [12] The Cartesian product of $G(V_1, E_1)$ and $H(V_2, E_2)$, $G \times H$ is the simple graph with $V_1 \times V_2$ as its vertex set and two vertices (u, v) and (u', v') are joined in $G \times H$ if and only if either $u = u'$ and v, v' are connected in H , or u, u' are connected in G and $v = v'$.

Definition 6 Let P_n be a path. Then the Cartesian product $P_n \times P_m$, where $m \leq n$, is called a square grid graph.

Theorem 7 A square grid permits a prime distance labeling for all integer $n \geq 4$.

The proof of Theorem 7 is direct from Theorem 1.

Definition 8 [12] The triangular grid graph T_n is the graph on vertices (i, j, k) with i, j, k being non-negative integers summing to n such that vertices are adjacent if the sum of absolute differences of the coordinates of two vertices is 2.

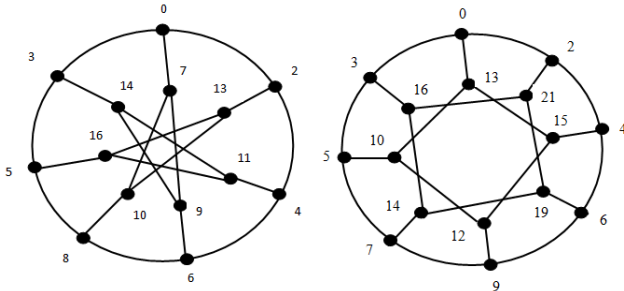


Figure 1A prime distance labeling of the GPG $P(7,3)$ and $P(8,2)$

Theorem 9 A triangular grid T_n permits a prime distance labeling for all positive integer $n \geq 1$.

Proof. Let $T_l, l \geq 3$ be the given triangular grid. We label the vertices of T_l on l levels as v_i^j for $1 \leq i \leq k$ and $1 \leq j \leq l$. See Figure 2. Define an injective function $f: V(T_l) \rightarrow Z$ as follows: let $f(v_1^1) = 0$ and $f(v_i^j) = f(v_{i-1}^{j-1}) + p_r, 2 \leq j \leq l, r < l$, where p_r are sufficiently large primes (not necessarily distinct). Then $f(v_i^j) = f(v_{i-1}^{j-1}) + 2$ for $2 \leq j \leq l$ and $2 \leq i \leq k$. It is now easy to see that f is the desired prime distance labeling of T_l .

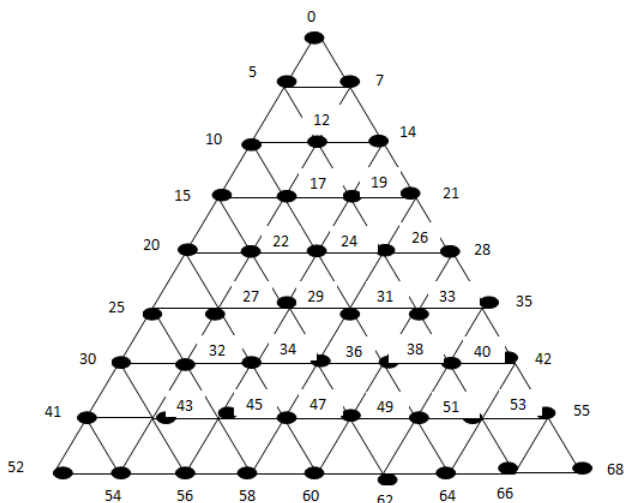


Figure 2A prime distance labeling of the triangular grid T_9

Definition 10 [3] Let H_1, H_2, \dots, H_k be k copies of a fixed graph H . The graph G obtained by adding an edge between H_i and H_{i+1} for $i = 1, 2, \dots, k - 1$ is called the path union of H .

Theorem 11 The graph obtained by the path union of finite number of copies of any prime distance graph admits a prime distance labelling.

Proof. Let H be the prime distance graph on n_1 vertices with a prime distance labeling h . Let P_{n_2} be a path on n_2 vertices. Let G be a graph obtained by the path union of $k (k \geq 2)$ copies of H . We label the vertices of the copies of H as $H_j: v_i^j$ where $1 \leq i \leq n_1$ and $1 \leq j \leq k$. Note that $|V(G)| = k|V(H)| + |E(G)| = k|E(H)| + |E(P_{n_2})|$. Define an injective function $f: V(G) \rightarrow Z$ as follows: let $f(H_1) = h(H_1)$. Let

$r_1 = \max\{f(H_1)\}$ and let p_{r_1} be any prime sufficiently larger than r_1 . Next $f(H_2) = f(H_1) + p_{r_1}$. Continuing the same process, let $r_{k-1} = \max\{f(H_{k-1})\}$ and let $p_{r_{k-1}}$ be any prime sufficiently larger than r_{k-1} . Next $f(H_k) = f(H_{k-1}) + p_{r_{k-1}}$. Thus f is the desired prime distance labeling of G .

Definition 12 [2] A Mongolian tent is a graph formed from $P_m \times P_n$ by introducing one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex.

Lemma 4 Let $M(m, n)$ be the mangolian tent. Then $M(2, n)$ permits a prime distance labeling for $1 \leq n \leq 10$.

Theorem 13 Let $M(m, n)$ be the Mongolian tent. Then $M(m, n)$ permits no prime distance labeling for any $m > 0$ and $n \geq 11$.

The proof of Theorem 13 follows easily from Theorem 3.

Theorem 14 The flower graph Fl_n admits no prime distance labeling for $n \geq 5$.

Theorem 15 The sun flower graph $V(n, s, t)$ does not admit a prime distance labeling for $n \geq 5$.

The proofs of Theorem 14 and Theorem 15 are direct from Theorem 2.

C. Distinct Prime Distance Labelling of cycles

In 2017, Parthiban et al. introduced the notion of distinct prime distance labeling of graphs [11]. A graph H is a distinct prime distance graph if there exists an injective function $f_d: V(H) \rightarrow Z$ such that for any two vertices s and t which are adjacent in H , the integer $|f_d(s) - f_d(t)|$ is a prime (distinct). So H is a distinct prime distance graph if and only if there exists a distinct prime distance labeling of H and f_d is called a distinct prime distance labeling of H . In this section, we prove that the cycles are distinct prime distance graphs.

Theorem 16 Let C_n be a cycle. Then C_n permits a distinct prime distance labeling for all $n \geq 3$.

Proof. Let C_n be a cycle on $n \geq 3$ vertices, namely, v_1, v_2, \dots, v_n . We divide the proof in to two cases.

Case 1: When $n \equiv 1 \pmod{2}$

The vertices of the cycles C_3 and C_5 can be labeled with the labels 0, 2, 5 and 0, 2, 5, 10, 17, respectively in the clockwise direction. So we take $n \geq 7$. Define an injective function $f: V(G) \rightarrow Z$ as follows: without loss of generality, let $f(v_1) = 0, f(v_2) = p_1, f(v_3) = p_1 + p_2, \dots, f(v_{n-1}) = p_1 + p_2 + \dots + p_{n-2}$, and $f(v_n) = p_1 + p_2 + \dots + p_{n-1}$ where $p_i \in P$ (the set of all primes) for $1 \leq i \leq n - 1$. Then the following two subcases arise.

Subcase 1: If $f(v_n)$ is a prime, then we are through.

Subcase 2: If $f(v_n)$ is not a prime, then add any sufficiently large prime greater than p_{n-1} to $f(v_{n-1})$ so as to make the label $f(v_n)$ a prime.

Case 2: When $n \equiv 0 \pmod{2}$

The vertices of the cycles C_4 and C_6 can be labeled with the labels 0, 7, 18, 31 and 0, 7, 18, 31, 48, 67, respectively in the clockwise direction. So we take $n \geq 8$. Define an one-to-one function $f: V(G) \rightarrow Z$ as follows: without loss of generality, let $f(v_1) = 0, f(v_2) = p_1, f(v_3) = f(v_2) + p_1, \dots, f(v_{n-1}) = f(v_{n-2}) + p_1 + p_2 + \dots + p_{n-3}$, and $f(v_n) = f(v_{n-1}) + p_1 + p_2 + \dots + p_{n-2}$ where $p_i \in P$ for $1 \leq i \leq n - 2$. Then the following two subcases arise.

Subcase 1: If $f(v_n)$ is a prime, then the proof is complete.

Subcase 2: If $f(v_n)$ is not a prime, then add any sufficiently large prime greater than p_{n-3} to $f(v_{n-1})$ so as to make the label $f(v_n)$ a prime. This completes the proof.

III.CONCLUSION

Prime distance labelling of the generalized Petersen graphs has been investigated, besides establishing the same for other classes of graphs. The distinct prime distance labelling of cycles is also obtained. We believe that the prime distance labelling may find applications in graph-based cryptography.

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