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On the Vertex Multiplication Graphs

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Abstract: *For any graph G, with vertex set $\{v_1, v_2, \ldots, v_p\}$ and a p-tuble of positive integers $(n_1, n_2, \ldots, n_p)$, the vertex multiplication graph $G_{n_1,n_2,\ldots,n_p}$ is defined as the graph with vertex set consists of n_i copies of each $v_i \in V(G)$, where the copies of v_i and v_j are adjacent in $G_{n_1,n_2,\ldots,n_p}$ if and only if the corresponding vertices v_i and v_j are adjacent in G. In this paper, we prove that the spectrum of $G_{n_1,n_2,\ldots,n_p}$ is same as that of spectrum of its quotient graph $\widetilde{G}_{n_1,n_2,\ldots,n_p}$ with additional zero eigenvalues with multiplicity $n-p$, where $n = n_1 + n_2 + \ldots + n_p$. Also we prove that the determinant of $\widetilde{G}_{\{n_1,n_2,\ldots,n_p\}}$ is minimum for $\widetilde{G}_{n-p+1,1,\ldots,1}$ and maximum for $\widetilde{G}_{\lceil\frac{n}{p}\rceil,\lceil\frac{n}{p}\rceil,\ldots,\lceil\frac{n}{p}\rceil,\lfloor\frac{n}{p}\rfloor,\lfloor\frac{n}{p}\rfloor,\ldots,\lfloor\frac{n}{p}\rfloor}$. Also we find distance- i spectrum of thorn graphs, G^{+k}, when G is connected r - regular graph with diameter 2.*

Keywords : vertex multiplication; eigen values; quotient graph.

I. INTRODUCTION

We follow the terminologies and notations from [6], [19]. Let G be a simple graph with vertex set $\{v_1, v_2, \ldots v_p\}$ and $(n_1, n_2, \ldots, n_p)$ be a $p-$ tuble of positive integers. Then the vertex multiplication graph $G_{n_1,n_2,\ldots,n_p}$ is defined as the graph with vertex set consists of n_i copies of each $v_i \in V(G)$, where the copies of v_i and v_j are adjacent in $G_{n_1,n_2,\ldots,n_p}$ if and only if the corresponding vertices v_i and v_j are adjacent in G. More precisely, $G_{n_1,n_2,\ldots,n_p}$ is the graph with vertex set V^* and edge set E^* such that $V^* = \cup_{i=1}^{p} V_i$ where $V_i = \{v_i^k \mid 1 \le k \le n_i\}$ and $E^* = \cup_{v_iv_j \in E(G)} \{v_i^l v_j^m \mid 1 \le l \le n_i, 1 \le m \le n_j\}$. Vertex multiplication graphs are studied by various authors in different contexts [14], [16], [19].

Let G be the path on 3 vertices, P_3, with vertex set $\{v_1, v_2, v_3\}$ as in **Fig.1**.

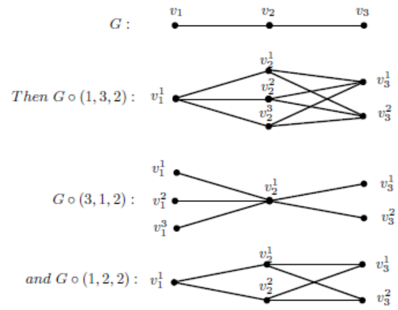

Fig.1. Graph G and its vertex multiplication

If $A(G)$ is the adjacency matrix of G and $\lambda_1, \lambda_2, \cdots, \lambda_p$ are the eigen values of $A(G)$, then the set $\{\lambda_1, \lambda_2, \cdots, \lambda_p\}$ is called as the spectrum of G, denoted by $Spec(G)$. Suppose $\theta_1, \theta_2, \cdots, \theta_k$ are the distinct eigen values of G and m_i is the multiplicity of $\theta_i (i = 1,2, \ldots, k)$, then thespectrum can be written as the multiset $\{[\theta_1]^{m_1}, [\theta_2]^{m_2}, \cdots, [\theta_k]^{m_k}\}$. The energy (G), of a graph G is defined as the sum of the absolute values of its eigen values, $\varepsilon(G) = \Sigma_{i=1}^{p} |\lambda_i|$. For more on spectrum and energy of graphs, we refer [3], [6], [10], [15].

In [18], the authors proved the following result.

Theorem:Consider complete multipartite graph $K_{n_1,n_2,\ldots,n_p}$ on $n = \sum_{i=1}^{p} n_i$ vertices and $n_1 \ge n_2 \ge \ldots \ge n_p > 0$. For fixed value of n, spectral radius and energy of $K_{n_1,n_2,\ldots,n_p}$ are minimum for complete split graph $K_{n-p+1,1,\ldots,1}$ denoted by $CS(n, p-1)$ and maximum for Turan graph $K_{\lceil\frac{n}{p}\rceil,\lceil\frac{n}{p}\rceil,\ldots,\lceil\frac{n}{p}\rceil,\lfloor\frac{n}{p}\rfloor,\lfloor\frac{n}{p}\rfloor,\ldots,\lfloor\frac{n}{p}\rfloor}$ denoted by $T(n, p)$.

Now the complete multipartite graph $K_{n_1,n_2,\ldots,n_p}$ can be seen as $K_{p\circ(n_1,n_2,\ldots,n_p)}$, vertex multiplication of complete graph K_p with the tuple $(n_1, n_2, \ldots, n_p)$ and the result can be modified as spectral radius and energy are minimum for the tuple $(n-p+1,1,\ldots,1)$ and maximum for the tuple $(\lceil\frac{n}{p}\rceil, \lceil\frac{n}{p}\rceil, \ldots, \lceil\frac{n}{p}\rceil, \lfloor\frac{n}{p}\rfloor, \lfloor\frac{n}{p}\rfloor, \ldots, \lfloor\frac{n}{p}\rfloor)$.

Motivated by the above result, we are interested in the study of spectrum and energy of $G \circ h$ for general graph G and finding the $p-$ tuples h for which energy and/or spectral radius are/is maximum or minimum. But, as the following examples show, unlike complete multipartite graph, the spectrum may vary for a general vertex multiplication graph $G \circ h$, if the coordinates of h are rearranged.

Example 1.3. Consider the graph $G = P_3$ depicted in **Fig.1.** Spectrum of its (non isomorphic) vertex multiplication graphs on 6 vertices are given below.

Graph	Edges	Spectrum
$G \circ (1,3,2)$	9	$\{[3]^1, [0]^4, [-3]^1\}$

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$G \circ (2,2,2)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$
$G \circ (1,2,3)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$
$G \circ (1,4,1)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$
$G \circ (4,1,1)$	5	$\{[\sqrt{5}]^1, [0]^4, [-\sqrt{5}]^1\}$

spectral radius and energy are maximum for $G\circ$ (1, 3, 2) and minimum for $G \circ$ (4, 1, 1).

Example 1.4. For the same graph G in the previous example, spectrum of its (non isomorphic) vertex multiplication graphs on 5 vertices are given below.

Graph	Edges	Spectrum
$G \circ (1,3,1)$	6	$\{[\sqrt{6}]^1, [0]^3, [-\sqrt{6}]^1\}$
$G \circ (1,2,2)$	6	$\{[\sqrt{6}]^1, [0]^3, [-\sqrt{6}]^1\}$
$G \circ (3,1,1)$	4	$\{[2]^1, [0]^3, [-2]^1\}$

spectral radius and energy are maximum for $G\circ$ (1, 3, 2) and minimum for $G \circ$ (4, 1, 1).

Example 1.5.

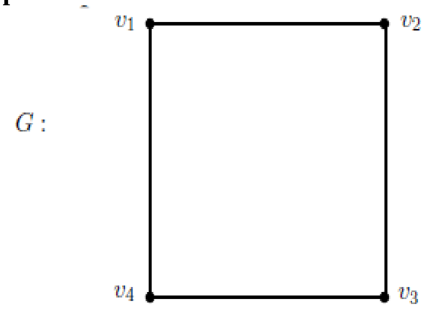

Fig.2 :G= C_4 with vertex set {v_1, v_2, v_3, v_4}

Let G= C_4 be as in **Fig.2.** spectrum of its (non isomorphic) vertex multiplication graphs on 6 vertices are given below.

Graph	Edges	Spectrum
$G \circ (2,2,1,1)$	9	$\{[3]^1, [0]^4, [-3]^1\}$
$G \circ (2,1,2,1)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$
$G \circ (3,1,1,1)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$

spectral radius and energy are maximum for $G\circ$(2, 2, 1, 1). $G\circ$(3, 1, 1, 1) and $G\circ$(2, 1, 2, 1) are cospectral and spectral radius and energy is minimum for them.

Example 1.6. Consider G= C_5-vertex multiplication graphs with 8 vertices. spectral radius is maximum for the graph $G\circ$(3, 2, 1, 1, 1) energy is maximum for the graph $G\circ$(2, 2, 2, 1, 1). spectral radius and energy are minimum for the graph $G\circ$ (3,1,2,1,1).

Anyway, in the next section we establish relations between spectrum of vertex multiplication graph and its quotient graph.

II. SPECTRUM OF VERTEX MULTIPLICATION GRAPHS

Wesaythatapartition$\pi$of $V(G)$ withcells$C_1,C_2,\ldots,C_p$is*equitable*ifthenumberof neighbours in$C_j$ofavertex$u$of$C_i$isaconstantb_{ij},independentof$u$.Equivalentlysubgraphof$G$inducedbyeachcellisregularandtheedges joininganytwodistinctcellsformasemiregularbipartitegraph.Now, The directed graph with the p cells of π asthe verticesandb_{ij}arcsfrom the $i^{th}$to the j^{th}cells of π iscalled*quotient*or *quotient graph* of G over π denoted byG/π.Thereforetheentriesoftheadjacencymatrixofthisquotientaregivenby$A(G/\pi)_{ij}=b_{ij}$.Quotientgraphshavebeenstudiedin[7].

Let G beanygraphon$p$vertices,withadjacencymatrix $A = [a_{ij}]$. Let $G_{n_1,n_2,\ldots,n_p}$be the vertex multiplication graph$G\circ$ (n_1, n_2, ..., n_p).Now the vertex partition π ={V_1, V_2, ...V_p} of $G_{n_1,n_2,\ldots,n_p}$is equitable, with $b_{ij}= n_j a_{ij}$for all i and j.The quotient graph of$G_{n_1,n_2,\ldots,n_p}/\pi$can be denoted as $\widetilde{G}_{n_1,n_2,\ldots,n_p}$. The adjacency matrix of quotient graph is$\widetilde{G}_{n_1,n_2,\ldots,n_p}$. is $A(\widetilde{G}_{n_1,n_2,\ldots,n_p})$=[$b_{ij}$]

$$= \begin{pmatrix} 0 & n_2a_{12} & n_3a_{13} & \cdots & n_pa_{1p} \\ n_1a_{21} & 0 & n_3a_{23} & & n_pa_{2p} \\ & \vdots & & \ddots & \vdots \\ n_1a_{p1} & n_2a_{p2} & n_3a_{p3} & \cdots & 0 \end{pmatrix} \qquad (1)$$

To find spectrum of vertex multiplication graph $G\circ$h, in a particular case $h = (t,t,t,\ldots t)$, we use tensor product of matrices.

Definition 2.1. If $A = [a_{ij}]$ is a matrix of order $m_1\times n_1$, and $B = [b_{ij}]$ is a matrix of order $m_2\times n_2$, then the tensor product of A and B, denoted by A⨂B, is a matrix of order $m_1m_2\times n_1n_2$, obtained by replacing each entry a_{ij} of A by the double array a_{ij}B.

Theorem 2.2. [6] If A is a matrix of order m with spectrum $\{\lambda_1, \lambda_2\ldots, \lambda_m\}$, and B a matrix of order n with spectrum $\{\mu_1,\mu_2,\ldots,\mu_n\}$, then the spectrum of A⨂B is $\{\lambda_i\mu_j{:}1\le i\le m;\ 1\le j\le n\}$.

Theorem 2.3. Let G be a graph on p vertices with spectrum $\{[\theta_1]^{m_1},[\theta_2]^{m_2},\ldots,[\theta_k]^{m_k}\}$.Then the spectrum of G∘h where h =(t,t,t,…,t) is $\{[t\theta_1]^{m_1},[t\theta_2]^{m_2},\ldots,[t\theta_k]^{m_k},[0]^{p(t-1)}\}$.

Proof. Since $h = (t,t,t,\ldots,t)$, $A(G\circ h)=A(G)\otimes J_{t\times t}$, where J is all one matrix. Now the proof follows from Theorem 2.2 and $Spec(J_{t\times t})=\{[t]^1,[0]^{t-1}\}$.

Corollary 2.4. $\varepsilon(G\circ h) = t\,\varepsilon(G)$

But it is hard to find the spectrum of $G\circ h$for general $h = (n_1, n_2, \ldots,n_p)$. But we can prove that,

Theorem 2.5. Foranygraph G ,andap-tuple of positive integers(n_1, n_2, ...,n_p),

$Spec(G_{n_1,n_2,\ldots,n_p})=Spec(\widetilde{G}_{n_1,n_2,\ldots,n_p}) \cup \{[0]^{n-p}\}$.

To prove this theorem, we use a generalization of joinoperation of two graphs, *H-join of a family of graphs*,defined by D. M. Cardesoet all in [5], as follows.

Definition 2.6. Considering the family of graph G_1,G_2...G_p, and the graph H, the $H-$ join ofG_1,G_2, ...G_p is the graph $G = \bigvee_H \{G_j, j \in V(H)\}$such that $V(G)=\bigcup_{r=1}^{p} V(G_r)$and

$$E(G) = \left(\bigcup_{r=1}^{p} E(G_r)\right) \cup \left(\bigcup_{rs\in E(H)} \{uv{:}\ u\in V(G_r), v\in V(G_s)\}\right)$$

and they established the following theorem about the spectrum of H-join graphs.

Theorem 2.7. [5] Consider the family $\mathcal{F}$ of p_j regular graphs, G_j of order n_j,for j∈{1,2,…,k} and a graph H such that V(H)={1,2,…,k}. Then

$Spec(\bigvee_H \mathcal{F})=(\bigcup_{j=1}^{p}(spec(G_j)-p_j)) \cup Spec(\widetilde{H})$, where$\widetilde{H}$ =[h_{ij}] is the k×k matrix defined by

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$$h_{ij} = \begin{cases} \sqrt{n_l n_q} & if\ i \neq j\ and\ lq \in E(H); \\ 0 & if\ i \neq j\ and\ lq \notin E(H). \end{cases}$$

and if i=j, h_{ij} is an element chosen form Spec(G_j).

Proof of theorem 2.5:

Let A=[a_{ij}] be the adjacency matrix of G. In the view of definition 2.6, the vertex multiplication graph$G_{n_1,n_2,\ldots,n_p} = G \circ (n_1, n_2, \ldots, n_p)$, can be seen as G- join of $\overline{K_{n_1}}$, $\overline{K_{n_2}}$,…,$\overline{K_{n_p}}$. Hence by applying Theorem 2.7, we get Spec($G_{n_1,n_2,\ldots,n_p}$)=($\cup_{j=1}^{p}(Spec(\overline{K_{n_j}})$-{0})) ∪ Spec ($\widetilde{H}$),

Where $\widetilde{H}$ is the following matrix:

$$=\begin{pmatrix} 0 & \sqrt{n_1n_2}a_{12} & \sqrt{n_1n_3}a_{13} & \cdots & \sqrt{n_1n_p}a_{1p} \\ \sqrt{n_2n_1}a_{21} & 0 & \sqrt{n_2n_3}a_{23} & & \sqrt{n_2n_p}a_{2p} \\ & \vdots & & \ddots & \vdots \\ \sqrt{n_pn_1}a_{p1} & \sqrt{n_pn_2}a_{p2} & \sqrt{n_pn_3}a_{p3} & \cdots & 0 \end{pmatrix}$$

$$=DAD, \text{ where } D=\begin{pmatrix} \sqrt{n_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{n_2} & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{n_p} \end{pmatrix}.$$

Now the proof follows from, equation (1) and DAD is similar to ADD.

Corollary 2.8. Rank of a graph, is the rank of the corresponding adjacency matrix. Then for any positive integers $n_1,n_2,\ldots,n_p$, rank (G)=rank ($G_{n_1,n_2,\ldots,n_p}$)=rank ($\tilde{G}_{n_1,n_2,\ldots,n_p}$).

Remark 2.9. In general, if G is a graph with equitable partition, and $\tilde{G}$ is its corresponding quotient graph, and $\phi(G,\lambda)$ and $\phi(\tilde{G},\lambda)$ are the respective characteristic polynomials then$\phi(\tilde{G},\lambda)$ divides $\phi(G,\lambda)$. But if G is any graph on p vertices and $h = (n_1, n_2, \ldots, n_p)$ is a p-tuple of positive integers, then

$$\phi\left(G_{n_1,n_2,\ldots,n_p},\lambda\right) = \lambda^{n-p}\phi\left(\tilde{G}_{n_1,n_2,\ldots,n_p},\lambda\right)$$

where $n = n_1 + n_2 + \cdots + n_p$

Theorem 2.10. If G is a graph on p vertices with spectrum $\{\ [\lambda]^1, [0]^{p-2}, [-\lambda]^1\ \}$,thenfor any positive integers $n_1,n_2,\ldots,n_p$,

Spec($G_{n_1,n_2,\ldots,n_p}$)= $\{[\sqrt{|E^*|}\ ]^1, [0]^{n-2}, [-\sqrt{|E^*|}\ ]^1\}$.

Proof. Now by Theorem 2.5, Spec ($G_{n_1,n_2,\ldots,n_p}$) will be of the form$\{[\theta]^1, [0]^{p-2}, [-\theta]^1\}$,. Then from the sachs theorem, -$\theta^2 = |E^*|$ and the proof follows.

III. DETERMINANT OF QUOTIENT GRAPH

Though we can not get a similar result as in [18] for the spectral parameters of vertex multiplication graphs, we obtain a similar result for determinant of its quotient graphs.

Note 3.1. Suppose two vertices, in a graph, have same neighbors, then their corresponding rows in the adjacency matrix are equal, and consequently the determinant of the graph will be zero. Hence, instead of G∘h we study the determinant of quotient graphs of G∘h.

Theorem 3.2. Let G be any graph on p verticesand$G_{n_1,n_2,\ldots,n_p}$be the vertex multiplication graph $G \circ (n_1, n_2, \ldots, n_p)$. Then the determinant of the quotient graph$\tilde{G}_{n_1,n_2,\ldots,n_p}$ is $(\prod_{i=1}^{p} n_i)\,|\,G\,|$.

Proof. If A(G)=[a_{ij}] and A($\tilde{G}_{n_1,n_2,\ldots,n_p}$)=[$b_{ij}$], then $b_{ij}=n_ja_{ij}$ for all i and j. Hence|A($\tilde{G}_{n_1,n_2,\ldots,n_p}$)|

$$= \begin{vmatrix} 0 & n_2a_{12} & n_3a_{13} & \cdots & n_pa_{1p} \\ n_1a_{21} & 0 & n_3a_{23} & & n_pa_{2p} \\ & \vdots & & \ddots & \vdots \\ n_1a_{p1} & n_2a_{p2} & n_3a_{p3} & \cdots & 0 \end{vmatrix}$$

=$(\prod_{i=1}^{p} n_i)$ |A(G)|

Result 3.3. If x and y are any two positive integers, with$x - y \geq 2$, then $(x - 1)(y + 1) > xy$.

Theorem 3.4. Consider the positive integers $n_1 \geq n_2 \geq \ldots \geq n_p >$ 0, for which $n = n_1 + n_2 + \ldots + n_p$. If n is fixed, then the product$\prod_{i=1}^{p} n_i$ is minimum if $n_1 = n - p + 1$ and n_i = 1for all $i > 1$ and maximum if $(n_1, n_2, \ldots, n_p) = (\lceil n/p \rceil, \lceil n/p \rceil, \ldots, \lceil n/p \rceil, \lfloor n/p \rfloor, \lfloor n/p \rfloor, \ldots, \lfloor n/p \rfloor)$.

*Proof.*Suppose, the product is minimum for $(n_1, n_2, \ldots, n_p)$ and $n_i \geq n_j \geq 2$, then $(n_i + 1) - (n_j - 1) \geq 2$ .Hence by the Result 3.3, $(n_i)(n_j) > (n_i + 1)(n_j - 1)$. Hence $n_1n_2 \ldots n_i \ldots n_j \ldots n_p > n_1n_2 \ldots (n_i + 1) \ldots (n_j - 1) \ldots n_p$,contradicting the choice of $(n_1, n_2, \ldots, n_p)$. Hence all integers $n_1, n_2, \ldots, n_p$are equal to one, except for one, which is equal to $n - p + 1$.

Suppose, the product is maximum for $(n_1, n_2, \ldots, n_p)$and for some $i, j, n_i - n_j \geq 2$, then by Result 3.3, $(n_i - 1)(n_j + 1) > (n_i)(n_j)$. Hence $n_1n_2 \ldots (n_i - 1) \ldots (n_j + 1) \ldots n_p > n_1n_2 \ldots n_i \ldots n_j \ldots n_p$, contradicting the choice of $(n_1, n_2, \ldots, n_p)$). Hence $|n_i - n_j| \leq 1$ for all $i \neq j$and so each n_i is equal to either $\left\lceil \frac{n}{p} \right\rceil$or$\left\lfloor \frac{n}{p} \right\rfloor$.

Theorem 3.5. Let $n_1+n_2+\ldots+n_p$ be equal to a fixed integern and G be any graph on p vertices, where p ≥ 2 . Then thedeterminant of$\tilde{G}_{n_1,n_2,\ldots,n_p}$is minimum for $\tilde{G}_{n-p+1,1,\ldots,1}$and maximum for $\tilde{G}_{\lceil n/p \rceil,\lceil n/p \rceil,\ldots,\lceil n/p \rceil,\lfloor n/p \rfloor,\lfloor n/p \rfloor,\ldots,\lfloor n/p \rfloor}$.

Proof. The proof follows by combining Theorem 3.2 and Theorem 3.4 □

IV. DISTANCE-iSPECTRUM OF GRAPHS

The *derived graph* of a simple graph G, denoted by G^+,is the graph having the same vertex set as G, in which two vertices are adjacent if and only if their distance in G is two . As the notion of *derived graph* representstheconcept of second electron affinity in inorganicchemistry, the *derived graph*,its *spectrum* and its*energy,* called as second stage spectrum and second stage energy respectively, were studied in [2] and continued in[11]. Similarly, the *radial graph* in[12] and *antipodalgraph in* [1] of a simple graph G,are the graphs having the same vertex as G,in which two vertices are adjacent if and only if their distance in G is *radius* of G and *diameter* of G respectively. A generalization of these concepts,known as *distance –i graphs* are studied in [4].

Definition4.1.*If G is a connected graph, distance- i graph of G, denoted by $T_i(G)$, is the graph with vertex set as of G and two vertices x and y are adjacent if they are at distance i in G* Motivated by second stage spectrum of graphs, we introduce distance –i spectrum of a graph G as the spectrum of its distance-$i$graph and denote it by $Spec_i(G)$.For i=1,we get usual spectrum ,and for i=2 , we get second stage spectrum.

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Consider a graph G with vertex set $\{v_{1,}v_2,\dots,v_p\}$ for every vertex $v_i$attach k pendant vertices. Then the resulting graph is known as Thorn graph or Thorny graph denoted by G^{+k}([8], [9], and [13]).

In[17], distance –*i* graphs of cycles, paths and hadamard graphs and their corresponding spectrum have been studied. Here we continue this on study Thorn graphs G^{+K} and find their distance –*i* spectrum, when G is connected r -regular graph with diameter 2.

Proposition 4.2. Let G be graph with vertex set $\{v_1,v_2,\dots,v_p\}$and $(n_1,n_2,\dots,n_p)$ be a p-tuple of positive integers . For $i\neq 2, T_i\left(G_{n_1,n_2,\dots,n_p}\right)_i = T_i(G)_{n_1,n_2,\dots,n_p}$ and for i=2, $T_2\left(G_{n_1,n_2,\dots,n_p}\right) = T_2(G)_{n_1,n_2,\dots,n_p}$ with additional edges,that form cliques on copies of same vertex. That is,replace each vertex v_i of $T_2(G)$ by K_{n_1} and all possible edges.

Proof. Let *u* and *v* be its two vertices of G and *u*' and *v*' be their Copies in $G_{n_1,n_2,\dots,n_p}$. Then $d(u',v') = d(u,v)$, if $u \neq v$, and $d(u',v') = 2$, if $u = v$. Hence the proof follows from the definition of distance–i graph.☐

Lemma 4.3. [6] *Let A, B, C and D be matrices and let* $M=\begin{bmatrix} A & B \\ C & D\end{bmatrix}$.*Then* $|M| = |AD - BC|$, *if A and C commutes*

Lemma 4.4.[6]Let Gbe a connected r-regular graph on p vertices with an adjacency matrix A having m distinct eigenvalues $\lambda_1 = r, \lambda_2, \dots, \lambda_m$. Then,thereexistsapolynomial $Q(x)=p\,\frac{(x-\lambda_2)(x-\lambda_3)\dots(x-\lambda_m}{(r-\lambda_2)(r-\lambda_3)\dots(r-\lambda_m)}$such that $Q(A) = J$ so that $Q(r) = p$ and $Q(\lambda_i) = 0 \forall \lambda_i \neq r$

Theorem 4.5:*Let* G *be a connected* r- *regular graph with diameter 2 on p vertices with spec(G)=*$\{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p\}$. *Then for* k ≥ 1, *distance*-i *spectrum of* G^{+k}*are given as below:*

$Spec_1(G^{+k})$ consists of

$\frac{1}{2}[\lambda_i \pm \sqrt{\lambda_i{}^2 + 4k}\,]$ with multiplicity 1 for i=1 to p;

0 with multiplicity $p(k-1)$.

$Spec_2(G^{+k})$ consists of

$\frac{1}{2}[k-1+p \pm \sqrt{(k-1+p)^2 - 4(pk-p-kr^2)}$with multiplicity 1;

$\frac{1}{2}[k-1 \pm \sqrt{(k-1)^2 + 4k\lambda_i{}^2}$ with multiplicity1for i=2 to p;

-1 with multiplicity $p(k-1)$.

$Spec_3(G^{+k})$ consists of

$\frac{1}{2}\left[kr \pm \sqrt{(kr)^2 + 4k(p-1-r^2)}\right]$with multiplicity 1;

$\frac{1}{2}\left[k\lambda_i \pm \sqrt{(k\lambda_i)^2 + 4k(1+\lambda_i)^2}\right]$with multiplicity1for i=2 to p;

0 with multiplicity $p(k-1)$.

$Spec_4(G^{+k})$ consists of

$k(p-1-r)$ with multiplicity 1;

$k(-1-\lambda_i)$with multiplicity1for i=2 to p;

0 with multiplicity pk.

Proof: First we consider the case k = 1, that is the graph G^{+1}. Let $\{v_1, v_2, \dots, v_p\}$ be the vertices of G and $\{v_{p+1}, v_{p+2}, \dots, v_{2p}\}$be the corresponding pendent vertices. Let A and $\bar{A}$ be the adjacency matrices of G and $\bar{G}$ respectively. Adjacency matrices of $T_i(G^{+1})$ are

$\begin{bmatrix} A & I \\ I & 0\end{bmatrix}, \begin{bmatrix} \bar{A} & A \\ A & 0\end{bmatrix}, \begin{bmatrix} 0 & \bar{A} \\ \bar{A} & A\end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & \bar{A}\end{bmatrix}$ for i = 1, 2, 3, 4 respectively. Now$G^{+k} = G^{+1}_{n1,n2,\dots n2p}$where $n_j = 1$ and $n_{p+j} = k$ for $j = 1, 2, \dots, p$. Hence from Theorems 2.5,2.7 and Proposition 4.2, spectrum of $T_i(G^{+k})$ are $\{[0]^{p(k-1)}\} \cup spec\begin{bmatrix} A & kI \\ I & 0\end{bmatrix}$, $\{[-1]^{p(k-1)}\} \cup spec\begin{bmatrix} \bar{A} & \sqrt{k}A \\ \sqrt{k}A & (k-1)I\end{bmatrix}$, $\{[0]^{p(k-1)}\} \cup spec\begin{bmatrix} 0 & k\bar{A} \\ \bar{A} & kA\end{bmatrix}$ and $\{[0]^{p(k-1)}\} \cup spec\begin{bmatrix} 0 & 0 \\ 0 & k\bar{A}\end{bmatrix}$for i = 1, 2, 3, 4 respectively. Now the proof follows from Lemmas 4.3 and 4.4.☐

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