

On the Vertex Multiplication Graphs

M. Saravanan, K. M. Kathiresan

Abstract: For any graph G , with vertex set $\{v_1, v_2, \dots, v_p\}$ and a p -tuple of positive integers (n_1, n_2, \dots, n_p) , the vertex multiplication graph G_{n_1, n_2, \dots, n_p} is defined as the graph with vertex set consists of n_i copies of each $v_i \in V(G)$, where the copies of v_i and v_j are adjacent in G_{n_1, n_2, \dots, n_p} if and only if the corresponding vertices v_i and v_j are adjacent in G . In this paper, we prove that the spectrum of G_{n_1, n_2, \dots, n_p} is same as that of spectrum of its quotient graph $\tilde{G}_{\{n_1, n_2, \dots, n_p\}}$ with additional zero eigenvalues with multiplicity $n - p$, where $n = n_1 + n_2 + \dots + n_p$. Also we prove that the determinant of $\tilde{G}_{\{n_1, n_2, \dots, n_p\}}$ is minimum for $\tilde{G}_{n-p+1, 1, \dots, 1}$ and maximum for $\tilde{G}_{\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor, \dots, \lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor}$. Also we find distance- i spectrum of thorn graphs, G^{+k} , when G is connected r -regular graph with diameter 2.

Keywords : vertex multiplication; eigen values; quotient graph.

I. INTRODUCTION

We follow the terminologies and notations from [6], [19]. Let G be a simple graph with vertex set $\{v_1, v_2, \dots, v_p\}$ and (n_1, n_2, \dots, n_p) be a p -tuple of positive integers. Then the vertex multiplication graph G_{n_1, n_2, \dots, n_p} is defined as the graph with vertex set consists of n_i copies of each $v_i \in V(G)$, where the copies of v_i and v_j are adjacent in G_{n_1, n_2, \dots, n_p} if and only if the corresponding vertices v_i and v_j are adjacent in G . More precisely, G_{n_1, n_2, \dots, n_p} is the graph with vertex set V^* and edge set E^* such that $V^* = \cup_{i=1}^p V_i$ where $V_i = \{v_i^k \mid 1 \leq k \leq n_i\}$ and $E^* = \cup_{v_i, v_j \in E(G)} \{v_i^l v_j^m \mid 1 \leq l \leq n_i, 1 \leq m \leq n_j\}$. Vertex multiplication graphs are studied by various authors in different contexts [14], [16], [19].

Let G be the path on 3 vertices, P_3 , with vertex set $\{v_1, v_2, v_3\}$ as in Fig.1.

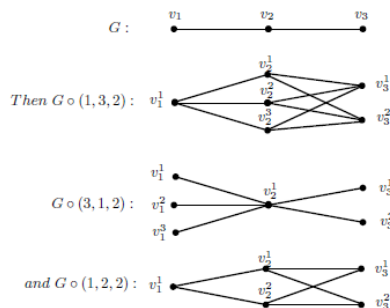


Fig.1. Graph G and its vertex multiplication

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If $A(G)$ is the adjacency matrix of G and $\lambda_1, \lambda_2, \dots, \lambda_p$ are the eigen values of $A(G)$, then the set $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$ is called as the spectrum of G , denoted by $Spec(G)$. Suppose $\theta_1, \theta_2, \dots, \theta_k$ are the distinct eigen values of G and m_i is the multiplicity of $\theta_i (i = 1, 2, \dots, k)$, then the spectrum can be written as the multiset $\{[\theta_1]^{m_1}, [\theta_2]^{m_2}, \dots, [\theta_k]^{m_k}\}$. The energy (G), of a graph G is defined as the sum of the absolute values of its eigen values, $\varepsilon(G) = \sum_{i=1}^p |\lambda_i|$. For more on spectrum and energy of graphs, we refer [3], [6], [10], [15].

In [18], the authors proved the following result.

Theorem: Consider complete multipartite graph K_{n_1, n_2, \dots, n_p} on $n = \sum_{i=1}^p n_i$ vertices and $n_1 \geq n_2 \geq \dots \geq n_p > 0$. For fixed value of n , spectral radius and energy of K_{n_1, n_2, \dots, n_p} are minimum for complete split graph $K_{n-p+1, 1, \dots, 1}$ denoted by $CS(n, p-1)$ and maximum for Turan graph $K_{\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor, \dots, \lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor}$ denoted by $T(n, p)$.

Now the complete multipartite graph K_{n_1, n_2, \dots, n_p} can be seen as $K_{p \circ (n_1, n_2, \dots, n_p)}$, vertex multiplication of complete graph K_p with the tuple (n_1, n_2, \dots, n_p) and the result can be modified as spectral radius and energy are minimum for the tuple $(n-p+1, 1, \dots, 1)$ and maximum for the tuple $(\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor, \dots, \lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor)$.

Motivated by the above result, we are interested in the study of spectrum and energy of $G \circ h$ for general graph G and finding the p -tuples h for which energy and/or spectral radius are/is maximum or minimum. But, as the following examples show, unlike complete multipartite graph, the spectrum may vary for a general vertex multiplication graph $G \circ h$, if the coordinates of h are rearranged.

Example 1.3. Consider the graph $G = P_3$ depicted in Fig.1. Spectrum of its (non isomorphic) vertex multiplication graphs on 6 vertices are given below.

Graph	Edges	Spectrum
$G \circ (1, 3, 2)$	9	$\{[3]^1, [0]^4, [-3]^1\}$
$G \circ (2, 2, 2)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$
$G \circ (1, 2, 3)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$
$G \circ (1, 4, 1)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$
$G \circ (4, 1, 1)$	5	$\{[\sqrt{5}]^1, [0]^4, [-\sqrt{5}]^1\}$

spectral radius and energy are maximum for $G \circ (1, 3, 2)$ and minimum for $G \circ (4, 1, 1)$.

Example 1.4. For the same graph G in the previous example, spectrum of its (non isomorphic) vertex multiplication graphs on 5 vertices are given below.

Graph	Edges	Spectrum
$G \circ (1,3,1)$	6	$\{[\sqrt{6}]^1, [0]^3, [-\sqrt{6}]^1\}$
$G \circ (1,2,2)$	6	$\{[\sqrt{6}]^1, [0]^3, [-\sqrt{6}]^1\}$
$G \circ (3,1,1)$	4	$\{[2]^1, [0]^3, [-2]^1\}$

spectral radius and energy are maximum for $G \circ (1, 3, 2)$ and minimum for $G \circ (4, 1, 1)$.

Example 1.5.

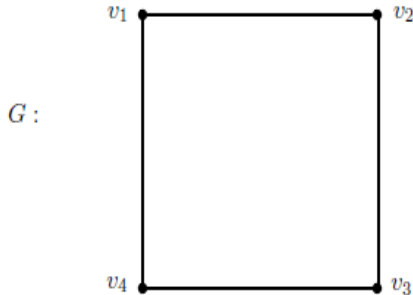


Fig.2 : $G=C_4$ with vertex set $\{v_1, v_2, v_3, v_4\}$

Let $G=C_4$ be as in Fig.2. spectrum of its (non isomorphic) vertex multiplication graphs on 6 vertices are given below.

Graph	Edges	Spectrum
$G \circ (2,2,1,1)$	9	$\{[3]^1, [0]^4, [-3]^1\}$
$G \circ (2,1,2,1)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$
$G \circ (3,1,1,1)$	8	$\{[2\sqrt{2}]^1, [0]^4, [-2\sqrt{2}]^1\}$

spectral radius and energy are maximum for $G \circ (2, 2, 1, 1)$. $G \circ (3, 1, 1, 1)$ and $G \circ (2, 1, 2, 1)$ are cospectral and spectral radius and energy is minimum for them.

Example 1.6. Consider $G=C_5$ -vertex multiplication graphs with 8 vertices. spectral radius is maximum for the graph $G \circ (3, 2, 1, 1, 1)$ energy is maximum for the graph $G \circ (2, 2, 2, 1, 1)$. spectral radius and energy are minimum for the graph $G \circ (3, 1, 2, 1, 1)$.

Anyway, in the next section we establish relations between spectrum of vertex multiplication graph and its quotient graph.

II. SPECTRUM OF VERTEX MULTIPLICATION GRAPHS

Wesaythatapartition π of $V(G)$ withcells C_1, C_2, \dots, C_p isequitablifthenumberof neighbours in C_j ofavertex u of C_i isaconstant b_{ij} ,independentof u .Equivalentlysubgraphof G inducedbyeachcellisregularandtheedges joininganytwodistinctcellsformasemiregularbipartitegraph. Now, The directed graph with the p cells of π as the vertices and b_{ij} arcs from the i^{th} to the j^{th} cells of π is called quotient or quotient graph of G over π denoted by G/π . Therefore the entries of the adjacency matrix of this quotient are given by $A(G/\pi)_{ij}=b_{ij}$. Quotient graphs have been studied in [7].

Let G be any graph on p vertices, with adjacency matrix $A = [a_{ij}]$. Let G_{n_1, n_2, \dots, n_p} be the vertex multiplication graph $G \circ (n_1, n_2, \dots, n_p)$. Now the vertex partition $\pi = \{V_1, V_2, \dots, V_p\}$ of G_{n_1, n_2, \dots, n_p} is equitable, with $b_{ij} = n_j a_{ij}$ for all i and j . The quotient graph of $G_{n_1, n_2, \dots, n_p}/\pi$ can be denoted as $\tilde{G}_{n_1, n_2, \dots, n_p}$. The adjacency matrix of quotient graph is $\tilde{A}(\tilde{G}_{n_1, n_2, \dots, n_p}) = [b_{ij}]$

$$= \begin{pmatrix} 0 & n_2 a_{12} & n_3 a_{13} & \dots & n_p a_{1p} \\ n_1 a_{21} & 0 & n_3 a_{23} & \dots & n_p a_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_1 a_{p1} & n_2 a_{p2} & n_3 a_{p3} & \dots & 0 \end{pmatrix} \quad (1)$$

To find spectrum of vertex multiplication graph $G \circ h$, in a particular case $h = (t, t, t, \dots, t)$, we use tensor product of matrices.

Definition 2.1. If $A = [a_{ij}]$ is a matrix of order $m_1 \times n_1$, and $B = [b_{ij}]$ is a matrix of order $m_2 \times n_2$, then the tensor product of A and B , denoted by $A \otimes B$, is a matrix of order $m_1 m_2 \times n_1 n_2$, obtained by replacing each entry a_{ij} of A by the double array $a_{ij} B$.

Theorem 2.2. [6] If A is a matrix of order m with spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$, and B a matrix of order n with spectrum $\{\mu_1, \mu_2, \dots, \mu_n\}$, then the spectrum of $A \otimes B$ is $\{\lambda_i \mu_j : 1 \leq i \leq m; 1 \leq j \leq n\}$.

Theorem 2.3. Let G be a graph on p vertices with spectrum $\{[\theta_1]^{m_1}, [\theta_2]^{m_2}, \dots, [\theta_k]^{m_k}\}$. Then the spectrum of $G \circ h$ where $h = (t, t, t, \dots, t)$ is $\{[t\theta_1]^{m_1}, [t\theta_2]^{m_2}, \dots, [t\theta_k]^{m_k}, [0]^{p(t-1)}\}$.

Proof. Since $h = (t, t, t, \dots, t)$, $A(G \circ h) = A(G) \otimes J_{t \times t}$, where J is all one matrix. Now the proof follows from Theorem 2.2 and $Spec(J_{t \times t}) = \{[t]^1, [0]^{t-1}\}$.

Corollary 2.4. $\varepsilon(G \circ h) = t \varepsilon(G)$

But it is hard to find the spectrum of $G \circ h$ for general $h = (n_1, n_2, \dots, n_p)$. But we can prove that,

Theorem 2.5. For any graph G , and a p -tuple of positive integers (n_1, n_2, \dots, n_p) , $Spec(G_{n_1, n_2, \dots, n_p}) = Spec(\tilde{G}_{n_1, n_2, \dots, n_p}) \cup \{[0]^{n-p}\}$.

To prove this theorem, we use a generalization of join operation of two graphs, H -join of a family of graphs, defined by D. M. Cardoso et al in [5], as follows.

Definition 2.6. Considering the family of graph G_1, G_2, \dots, G_p , and the graph H , the H -join of G_1, G_2, \dots, G_p is the graph $G = V_H \{G_j, j \in V(H)\}$ such that $V(G) = \cup_{r=1}^p V(G_r)$ and

$$E(G) = \left(\bigcup_{r=1}^p E(G_r) \right) \cup \left(\bigcup_{rs \in E(H)} \{uv : u \in V(G_r), v \in V(G_s)\} \right)$$

and they established the following theorem about the spectrum of H -join graphs.

Theorem 2.7. [5] Consider the family \mathcal{F} of p_j regular graphs, G_j of order n_j , for $j \in \{1, 2, \dots, k\}$ and a graph H such that $V(H) = \{1, 2, \dots, k\}$. Then

$Spec(V_H \mathcal{F}) = (\cup_{j=1}^p (spec(G_j) - p_j)) \cup Spec(\tilde{H})$, where $\tilde{H} = [h_{ij}]$ is the $k \times k$ matrix defined by

$$h_{ij} = \begin{cases} \sqrt{n_i n_j} & \text{if } i \neq j \text{ and } ij \in E(H); \\ 0 & \text{if } i \neq j \text{ and } ij \notin E(H). \end{cases}$$

and if $i=j$, h_{ij} is an element chosen from $Spec(G_j)$.



Proof of theorem 2.5:

Let $A=[a_{ij}]$ be the adjacency matrix of G . In the view of definition 2.6, the vertex multiplication graph $G_{n_1, n_2, \dots, n_p} = G \circ (n_1, n_2, \dots, n_p)$, can be seen as G - join of $\overline{K_{n_1}}, \overline{K_{n_2}}, \dots, \overline{K_{n_p}}$. Hence by applying Theorem 2.7, we get $\text{Spec}(G_{n_1, n_2, \dots, n_p}) = (\cup_{j=1}^p (\text{Spec}(\overline{K_{n_j}}) - \{0\})) \cup \text{Spec}(\tilde{H})$,

Where \tilde{H} is the following matrix:

$$= \begin{pmatrix} 0 & \sqrt{n_1 n_2} a_{12} & \sqrt{n_1 n_3} a_{13} & \dots & \sqrt{n_1 n_p} a_{1p} \\ \sqrt{n_2 n_1} a_{21} & 0 & \sqrt{n_2 n_3} a_{23} & \dots & \sqrt{n_2 n_p} a_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{n_p n_1} a_{p1} & \sqrt{n_p n_2} a_{p2} & \sqrt{n_p n_3} a_{p3} & \dots & 0 \end{pmatrix}$$

$$= \text{DAD}, \text{ where } D = \begin{pmatrix} \sqrt{n_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{n_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{n_p} \end{pmatrix}$$

Now the proof follows from, equation (1) and DAD is similar to ADD.

Corollary 2.8. Rank of a graph, is the rank of the corresponding adjacency matrix. Then for any positive integers n_1, n_2, \dots, n_p , $\text{rank}(G) = \text{rank}(G_{n_1, n_2, \dots, n_p}) = \text{rank}(\tilde{G}_{n_1, n_2, \dots, n_p})$.

Remark 2.9. In general, if G is a graph with equitable partition, and \tilde{G} is its corresponding quotient graph, and $\phi(G, \lambda)$ and $\phi(\tilde{G}, \lambda)$ are the respective characteristic polynomials then $\phi(\tilde{G}, \lambda)$ divides $\phi(G, \lambda)$. But if G is any graph on p vertices and $h = (n_1, n_2, \dots, n_p)$ is a p -tuple of positive integers, then

$$\phi(G_{n_1, n_2, \dots, n_p}, \lambda) = \lambda^{n-p} \phi(\tilde{G}_{n_1, n_2, \dots, n_p}, \lambda)$$

where $n = n_1 + n_2 + \dots + n_p$

Theorem 2.10. If G is a graph on p vertices with spectrum $\{[\lambda]^1, [0]^{p-2}, [-\lambda]^1\}$, then for any positive integers n_1, n_2, \dots, n_p ,

$$\text{Spec}(G_{n_1, n_2, \dots, n_p}) = \{[\sqrt{|E^*|}]^1, [0]^{n-2}, [-\sqrt{|E^*|}]^1\}.$$

Proof. Now by Theorem 2.5, $\text{Spec}(G_{n_1, n_2, \dots, n_p})$ will be of the form $\{[\theta]^1, [0]^{p-2}, [-\theta]^1\}$. Then from the Sachs theorem, $-\theta^2 = |E^*|$ and the proof follows.

III. DETERMINANT OF QUOTIENT GRAPH

Though we can not get a similar result as in [18] for the spectral parameters of vertex multiplication graphs, we obtain a similar result for determinant of its quotient graphs.

Note 3.1. Suppose two vertices, in a graph, have same neighbors, then their corresponding rows in the adjacency matrix are equal, and consequently the determinant of the graph will be zero. Hence, instead of $G \circ h$ we study the determinant of quotient graphs of $G \circ h$.

Theorem 3.2. Let G be any graph on p vertices and G_{n_1, n_2, \dots, n_p} be the vertex multiplication graph $G \circ (n_1, n_2, \dots, n_p)$. Then the determinant of the quotient graph $\tilde{G}_{n_1, n_2, \dots, n_p}$ is $(\prod_{i=1}^p n_i) |G|$.

Proof. If $A(G)=[a_{ij}]$ and $A(\tilde{G}_{n_1, n_2, \dots, n_p})=[b_{ij}]$, then $b_{ij}=n_j a_{ij}$ for all i and j . Hence $|A(\tilde{G}_{n_1, n_2, \dots, n_p})|$

$$= \begin{vmatrix} 0 & n_2 a_{12} & n_3 a_{13} & \dots & n_p a_{1p} \\ n_1 a_{21} & 0 & n_3 a_{23} & \dots & n_p a_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_1 a_{p1} & n_2 a_{p2} & n_3 a_{p3} & \dots & 0 \end{vmatrix}$$

$$= (\prod_{i=1}^p n_i) |A(G)|$$

Result 3.3. If x and y are any two positive integers, with $x - y \geq 2$, then $(x - 1)(y + 1) > xy$.

Theorem 3.4. Consider the positive integers $n_1 \geq n_2 \geq \dots \geq n_p > 0$, for which $n = n_1 + n_2 + \dots + n_p$. If n is fixed, then the product $\prod_{i=1}^p n_i$ is minimum if $n_1 = n - p + 1$ and $n_i = 1$ for all $i > 1$ and maximum if $(n_1, n_2, \dots, n_p) = ([n/p], [n/p], \dots, n/p, n/p, n/p, \dots, n/p)$.

Proof. Suppose, the product is minimum for (n_1, n_2, \dots, n_p) and $n_i \geq n_j \geq 2$, then $(n_i + 1) - (n_j - 1) \geq 2$. Hence by the Result 3.3, $(n_i)(n_j) > (n_i + 1)(n_j - 1)$. Hence $n_1 n_2 \dots n_i \dots n_j \dots n_p > n_1 n_2 \dots (n_i + 1) \dots (n_j - 1) \dots n_p$, contradicting the choice of (n_1, n_2, \dots, n_p) . Hence all integers n_1, n_2, \dots, n_p are equal to one, except for one, which is equal to $n - p + 1$.

Suppose, the product is maximum for (n_1, n_2, \dots, n_p) and for some $i, j, n_i - n_j \geq 2$, then by Result 3.3, $(n_i - 1)(n_j + 1) > (n_i)(n_j)$. Hence $n_1 n_2 \dots (n_i - 1) \dots (n_j + 1) \dots n_p > n_1 n_2 \dots n_i \dots n_j \dots n_p$, contradicting the choice of (n_1, n_2, \dots, n_p) . Hence $|n_i - n_j| \leq 1$ for all $i \neq j$ and so each n_i is equal to either $\lfloor \frac{n}{p} \rfloor$ or $\lceil \frac{n}{p} \rceil$.

Theorem 3.5. Let $n_1 + n_2 + \dots + n_p$ be equal to a fixed integer and G be any graph on p vertices, where $p \geq 2$. Then the determinant of $\tilde{G}_{n_1, n_2, \dots, n_p}$ is minimum for $\tilde{G}_{n-p+1, 1, \dots, 1}$ and maximum for $\tilde{G}_{\lfloor n/p \rfloor, \lfloor n/p \rfloor, \dots, \lfloor n/p \rfloor, \lfloor n/p \rfloor, \lfloor n/p \rfloor, \dots, \lfloor n/p \rfloor}$.

Proof. The proof follows by combining Theorem 3.2 and Theorem 3.4 \square

IV. DISTANCE-*i*SPECTRUM OF GRAPHS

The *derived graph* of a simple graph G , denoted by G^+ , is the graph having the same vertex set as G , in which two vertices are adjacent if and only if their distance in G is two. As the notion of *derived graph* represents the concept of second electron affinity in inorganic chemistry, the *derived graph*, its *spectrum* and its *energy*, called as second stage spectrum and second stage energy respectively, were studied in [2] and continued in [11]. Similarly, the *radial graph* in [12] and *antipodal graph* in [1] of a simple graph G , are the graphs having the same vertex set as G , in which two vertices are adjacent if and only if their distance in G is *radius* of G and *diameter* of G respectively. A generalization of these concepts, known as *distance -i graphs* are studied in [4].

Definition 4.1. If G is a connected graph, *distance -i graph* of G , denoted by $T_i(G)$, is the graph with vertex set as of G and two vertices x and y are adjacent if they are at distance i in G .

Motivated by second stage spectrum of graphs, we introduce distance -i spectrum of a graph G as the spectrum of its distance- i graph and denote it by $\text{Spec}_i(G)$. For $i=1$, we get usual spectrum, and for $i=2$, we get second stage spectrum.



Consider a graph G with vertex set $\{v_1, v_2, \dots, v_p\}$ for every vertex v_i attach k pendant vertices. Then the resulting graph is known as Thorn graph or Thorny graph denoted by G^{+k} [8], [9], and [13].

In [17], distance $-i$ graphs of cycles, paths and hadamard graphs and their corresponding spectrum have been studied. Here we continue this on study Thorn graphs G^{+k} and find their distance $-i$ spectrum, when G is connected r -regular graph with diameter 2.

Proposition 4.2. Let G be graph with vertex set $\{v_1, v_2, \dots, v_p\}$ and (n_1, n_2, \dots, n_p) be a p -tuple of positive integers. For $i \neq 2, T_i(G_{n_1, n_2, \dots, n_p}) = T_i(G)_{n_1, n_2, \dots, n_p}$ and for $i=2, T_2(G_{n_1, n_2, \dots, n_p}) = T_2(G)_{n_1, n_2, \dots, n_p}$ with additional edges, that form cliques on copies of same vertex. That is, replace each vertex v_i of $T_2(G)$ by K_{n_i} and all possible edges.

Proof. Let u and v be its two vertices of G and u' and v' be their Copies in G_{n_1, n_2, \dots, n_p} . Then $d(u', v') = d(u, v)$, if $u \neq v$, and $d(u', v') = 2$, if $u = v$. Hence the proof follows from the definition of distance $-i$ graph. \square

Lemma 4.3. [6] Let A, B, C and D be matrices and let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then $|M| = |AD - BC|$, if A and C commutes

Lemma 4.4. [6] Let G be a connected r -regular graph on p vertices with an adjacency matrix A having m distinct eigenvalues $\lambda_1 = r, \lambda_2, \dots, \lambda_m$. Then, there exists a polynomial $Q(x) = p \frac{(x-\lambda_2)(x-\lambda_3)\dots(x-\lambda_m)}{(r-\lambda_2)(r-\lambda_3)\dots(r-\lambda_m)}$ such that $Q(A) = J$ so that $Q(r) = p$ and $Q(\lambda_i) = 0 \forall \lambda_i \neq r$

Theorem 4.5: Let G be a connected r -regular graph with diameter 2 on p vertices with $spec(G) = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p\}$. Then for $k \geq 1$, distance $-i$ spectrum of G^{+k} are given as below:

$Spec_1(G^{+k})$ consists of

$$\frac{1}{2}[\lambda_i \pm \sqrt{\lambda_i^2 + 4k}] \text{ with multiplicity } 1 \text{ for } i=1 \text{ to } p;$$

$$0 \text{ with multiplicity } p(k-1).$$

$Spec_2(G^{+k})$ consists of

$$\frac{1}{2}[k-1+p \pm \sqrt{(k-1+p)^2 - 4(pk-p-kr^2)}] \text{ with multiplicity } 1;$$

$$\frac{1}{2}[k-1 \pm \sqrt{(k-1)^2 + 4k\lambda_i^2}] \text{ with multiplicity } 1 \text{ for } i=2 \text{ to } p;$$

$$-1 \text{ with multiplicity } p(k-1).$$

$Spec_3(G^{+k})$ consists of

$$\frac{1}{2}[kr \pm \sqrt{(kr)^2 + 4k(p-1-r^2)}] \text{ with multiplicity } 1;$$

$$\frac{1}{2}[k\lambda_i \pm \sqrt{(k\lambda_i)^2 + 4k(1+\lambda_i^2)}] \text{ with multiplicity } 1 \text{ for } i=2 \text{ to } p;$$

$$0 \text{ with multiplicity } p(k-1).$$

$Spec_4(G^{+k})$ consists of

$$k(p-1-r) \text{ with multiplicity } 1;$$

$$k(-1-\lambda_i) \text{ with multiplicity } 1 \text{ for } i=2 \text{ to } p;$$

$$0 \text{ with multiplicity } pk.$$

Proof: First we consider the case $k = 1$, that is the graph G^{+1} . Let $\{v_1, v_2, \dots, v_p\}$ be the vertices of G and $\{v_{p+1}, v_{p+2}, \dots, v_{2p}\}$ be the corresponding pendent vertices. Let A and \bar{A} be the adjacency matrices of G and \bar{G} respectively. Adjacency matrices of $T_i(G^{+1})$ are

$$\begin{bmatrix} A & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} \bar{A} & A \\ A & 0 \end{bmatrix}, \begin{bmatrix} 0 & \bar{A} \\ \bar{A} & A \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & \bar{A} \end{bmatrix} \text{ for } i = 1, 2, 3, 4 \text{ respectively. Now } G^{+k} = G_{n_1, n_2, \dots, n_{2p}}^{+k} \text{ where } n_j = 1 \text{ and } n_{p+j} = k \text{ for } j = 1, 2, \dots, p. \text{ Hence from Theorems 2.5, 2.7 and Proposition 4.2, spectrum of } T_i(G^{+k}) \text{ are } \{[0]^{p(k-1)}\} \cup spec \begin{bmatrix} A & kI \\ I & 0 \end{bmatrix}, \{[-1]^{p(k-1)}\} \cup spec \begin{bmatrix} \bar{A} & \sqrt{kA} \\ \sqrt{kA} & (k-1)I \end{bmatrix}, \{[0]^{p(k-1)}\} \cup spec \begin{bmatrix} 0 & k\bar{A} \\ \bar{A} & kA \end{bmatrix} \text{ and } \{[0]^{p(k-1)}\} \cup spec \begin{bmatrix} 0 & 0 \\ 0 & k\bar{A} \end{bmatrix} \text{ for } i = 1, 2, 3, 4 \text{ respectively. Now the proof follows from Lemmas 4.3 and 4.4. } \square$$

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REFERENCES

1. R. Aravamuthan, B. Rajendran, On antipodal graphs, *Discrete Mathematics*, 49(1984) 193-195.
2. S. K. Ayyaswamy, S. Balachandran, I. Gutman, On Second stage spectrum and Energy of a graph, *Kragujevac Journal of Mathematics*, 34(2010) 139-146.
3. R. Balakrishnan, The energy of a graph, *Linear Algebra and its Applications*, 387(2004) 287-295.
4. A. E. Brouwer, A. M. Cohen, A. Neumair, *Distance Regular graphs*, Springer - verlag, New York, 1989.
5. D.M. Cardoso, M.A.A. de Freitas, E.A. Martins, M. Robbiano, Spectra of graphs obtained by a generalization of the join graph operation, *Discrete Mathematics*, 313 (2013) 733-741.
6. D. Cvetkovic, M. Doob, H. Sachs, *Spectra of Graphs : Theory and Application*, Academic Press, New York, 1980.
7. C. Godsil, G. Royle, *Algebraic graph theory*, Springer - verlag, New York, Inc, 2001.
8. E. I. Gutman, Generalizations of recurrence relation for the characteristic polynomial of trees, *Publ. Inst. Math. (Beograd)* 63 (1998) 31-36.
9. I. Gutman, Distance in thorny graph, *Publ. Inst. Math. (Beograd)* 21(1977) 75-80
10. I. Gutman, in *The Energy of a Graph: Old and New Results*, in: A. Beten A. Kohnert, R. Laue, A. Wassermann (eds.) *Algebraic Combinatorics and Applications*, Springer - verlag, Berlin 2001, 196-211
11. S. R. Jog, S. B. Hande, I. Gutman, S. B. Bozkurt, Derived graphs of some graphs, *Kragujevac Journal of Mathematics*, 36 (2012) 309-314
12. K.M. Kathiresan, G. Marimuthu, A study on radial graphs, *Ars Combinatoria*, 96 (2010) 353-360.
13. D. J. Klein, T. Doslic and D. Bonchev, Vertex-weightings for distance moments and thorny graphs, *Discr. Appl. Math.*, 155 (2007), 2294-2302.
14. K.M. Koh, E.G. Tay, On optimal orientations of G vertex-multiplications, *Discrete Mathematics*, 219 (2000) 153-171.
15. X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.
16. K.L. Ng, K.M. Koh, On optimal orientation of cycle vertex multiplications, *Discrete Mathematics* 297 (2005) 104-118.
17. M. Saravanan, K.M. Kathiresan, On the energy of Hadamard Graph, *Bull. Inst. Combin. Appl.* 76(2016), 16-28.
18. D. Stevanovic, I. Gutman, M.U. Rehman, On spectral radius and energy of complete multipartite graphs, *Ars Mathematica Contemporanea* 9 (2015) 109-113.
19. D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, New Jersey, 2000.



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