On the Vertex Multiplication Graphs

M. Saravanan, K. M. Kathiresan

Abstract: For any graph $G$, with vertex set $\{v_1, v_2, \ldots, v_p\}$ and a $p$-tuple of positive integers $(n_1, n_2, \ldots, n_p)$, the vertex multiplication graph $G_{n_1, n_2, \ldots, n_p}$ is defined as the graph with vertex set consists of $n_i$ copies of each $v_i \in V(G)$, where the copies of $v_i$ and $v_j$ are adjacent in $G_{n_1, n_2, \ldots, n_p}$ if and only if the corresponding vertices $v_i$ and $v_j$ are adjacent in $G$. In this paper, we prove that the spectrum of $G_{n_1, n_2, \ldots, n_p}$ is same as that of spectrum of its quotient graph with additional zero eigenvalues with multiplicity $n - p$, where $n = n_1 + n_2 + \ldots + n_p$. Also we prove that the determinant of $G_{n_1, n_2, \ldots, n_p}$ is minimum for $\delta_{n-p+1,1,1}$ and maximum for $\delta_{1,2,3,\ldots,1}$, when $G$ is connected $r$-regular graph with diameter 2.

Keywords: vertex multiplication; eigen values; quotient graph.

I. INTRODUCTION

We follow the terminologies and notations from [6], [19].

Let $G$ be a simple graph with vertex set $\{v_1, v_2, \ldots, v_p\}$ and $(n_1, n_2, \ldots, n_p)$ be a $p$-tuple of positive integers. Then the vertex multiplication graph $G_{n_1, n_2, \ldots, n_p}$ is defined as the graph with vertex set consists of $n_i$ copies of each $v_i \in V(G)$, where the copies of $v_i$ and $v_j$ are adjacent in $G_{n_1, n_2, \ldots, n_p}$ if and only if the corresponding vertices $v_i$ and $v_j$ are adjacent in $G$. More precisely, $G_{n_1, n_2, \ldots, n_p}$ is the graph with vertex set $V^*$ and edge set $E^*$ such that $V^* = \bigcup_{0 \leq i \leq p} V_{i}$, where $V_i = \{v_i^0, v_i^1, \ldots, v_i^{n_i-1}\}$. Vertex multiplication graphs are studied by various authors in different contexts [14], [16], [19].

Let $G$ be the path on 3 vertices, $P_3$, with vertex set $\{v_1, v_2, v_3\}$ as in Fig.1.

![Fig.1. Graph G and its vertex multiplication](image)

If $A(G)$ is the adjacency matrix of $G$ and $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigen values of $A(G)$, then the set $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ is called as the spectrum of $G$, denoted by $Spec(G)$.

In [18], the authors proved the following result.

**Theorem:** Consider complete multipartite graph $K_{n_1, n_2, \ldots, n_p}$ on $n = \sum_{i=1}^{p} n_i$ vertices and $n_1 \geq n_2 \geq \ldots \geq n_p > 0$.

For fixed value of $n$, spectral radius and energy of $K_{n_1, n_2, \ldots, n_p}$ are minimum for complete split graph $K_{n-p+1,1,1}$ and maximum for Turan graph $K_{p,1,1,\ldots,1}$ denoted by $T(n,p)$.

In the previous example, we are interested in the study of spectrum and energy of $G \circ h$ for general graph $G$ and finding the $p$-tuples of $h$ for which energy and/or spectral radius are/is maximum or minimum. But, as the following examples show, unlike complete multipartite graph, the spectrum may vary for a general vertex multiplication graph $G \circ h$, if the coordinates of $h$ are rearranged.

**Example 1.3.** Consider the graph $G = P_3$, depicted in Fig.1. Spectrum of its (non isomorphic) vertex multiplication graphs on 6 vertices are given below.

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Example 1.5.

Let \( G = C_4 \) be as in Fig.2, spectrum of its (non isomorphic) vertex multiplication graphs on 6 vertices are given below.

Example 1.6. Consider \( G = C_3 \)-vertex multiplication graphs with 8 vertices. spectral radius is maximum for the graph \( G = (2, 2, 2, 1, 1) \) and \( G = (2, 1, 2, 1) \) are cospectral and spectral radius and energy is minimum for them.

II. SPECTRUM OF VERTEX MULTIPLICATION GRAPHS

Wesaythatapartition of \( V(G) \) with cells \( C_1, C_2, ..., C_n \) is equitable if all neighbours in \( C_i \) of a vertex \( v \) in \( C_i \) is constant and independent of \( u \). Equivalently, subgraph of a graph \( G \) induced by each cell is regular and the edges joining any two distinct cells form an 

\[ \text{Spec}(G) = \{\lambda_1, \lambda_2, ..., \lambda_n\} \]

spectral radius and energy are maximum for \( G = (1, 3, 2) \) and minimum for \( G = (4, 1, 1) \).

The adjacency matrix of quotient graph of graph \( G \) is denoted \( A(G) \).

\[ A(G) = (a_{ij}) \]

where \( a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is a neighbour of } v_j \text{ in } G \\ 0 & \text{otherwise} \end{cases} \)

The directed graph with the arcs from the vertex \( v_i \) to the vertex \( v_j \) is called 

\[ \text{Spec}(G) = \{\lambda_1, \lambda_2, ..., \lambda_n\} \]

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Anyway, in the next section we establish relations between spectrum of vertex multiplication graph and its quotient graph.

To find spectrum of vertex multiplication graph \( G \circ h \), in a particular case \( h = (t, t, t, ..., t) \), we use tensor product of matrices.

**Definition 2.1**. If \( A = [a_{ij}] \) is a matrix of order \( m \times n \), and \( B = [b_{ij}] \) is a matrix of order \( p \times q \), then the tensor product of \( A \) and \( B \), denoted by \( A \otimes B \), is a matrix of order \( mp \times nq \), obtained by replacing each entry \( a_{ij} \) of \( A \) by the double array \( a_{ij} \).

**Theorem 2.2**. [6] Let \( G \) be a graph on \( p \) vertices and \( K \) be any graph on \( q \) vertices with spectrum \( \{k_1, k_2, ..., k_q\} \). Then the spectrum of \( G \otimes K \) is \( \{tk_1, tk_2, ..., tk_q\} \).

**Proof.** Let \( h = (t, t, t, ..., t) \) be the vertex multiplication graph of any graph \( G \). Consider the family \( G = \{G_i\}_{i=1}^p \) of all \( G \) and the graph \( H \), the \( - \) join of \( G \) is the graph 

\[ G = \bigvee_{i=1}^p G_i \]

such that \( V(G) = \bigcup_{i=1}^p V(G_i) \) and 

\[ E(G) = \bigcup_{i=1}^p E(G_i) \cup \bigcup_{i<j} \{uv : u \in V(G_i), v \in V(G_j)\} \]

and they established the following theorem about the spectrum of \( - \)-join graphs.

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Proof of theorem 2.5:
Let A=[a_{ij}] be the adjacency matrix of G. In the view of definition 2.6, the vertex multiplication graph $G_{n_1,n_2,...,n_p}$ is equal to $G^p$ where \( n_1, n_2, ..., n_p \), can be seen as G- join of $K_{n_1}$, $K_{n_2}$, ..., $K_{n_p}$. Hence, by applying Theorem 2.7, we get $Spec(G_{n_1,n_2,...,n_p})=(\bigcup_{j=1}^{p}Spec(K_{n_j})) \cup Spec(\overline{H})$, where $H$ is the following matrix:

\[
\begin{pmatrix}
0 & \sqrt{n_1}a_{12} & \sqrt{n_1}a_{13} & \ldots & \sqrt{n_1}a_{1p} \\
-\sqrt{n_2}a_{21} & 0 & \sqrt{n_2}a_{23} & \ldots & \sqrt{n_2}a_{2p} \\
-\sqrt{n_3}a_{31} & -\sqrt{n_3}a_{32} & 0 & \ldots & \sqrt{n_3}a_{3p} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\sqrt{n_p}a_{p1} & -\sqrt{n_p}a_{p2} & -\sqrt{n_p}a_{p3} & \ldots & 0 \\
\end{pmatrix}
\]

=DAD, where $D=\begin{pmatrix}
\sqrt{n_1} & 0 & 0 & \ldots & 0 \\
0 & \sqrt{n_2} & 0 & \ldots & 0 \\
0 & 0 & \sqrt{n_3} & \ldots & 0 \\
0 & 0 & 0 & \ldots & \sqrt{n_p} \\
\end{pmatrix}$.

Now the proof follows from, equation (1) and DAD is similar to ADD.

Corollary 2.8. Rank of a graph, is the rank of the corresponding adjacency matrix. Then for any positive integers $n_1,n_2,...,n_p$, rank $(G_{n_1,n_2,...,n_p})=rank (G_{n_1,n_2,...,n_p})$.

Remark 2.9. In general, if G is a graph with equitable partition, and $\overline{G}$ is its corresponding quotient graph, and $\phi(G,\lambda)$ and $\phi(\overline{G},\lambda)$ are the respective characteristic polynomials then $\phi(G,\lambda)$ divides $\phi(\overline{G},\lambda)$. But if G is any graph on $p$ vertices and $h=(n_2, n_2, ..., n_p)$ is a p-tuple of positive integers, then

$\phi(G_{n_1,n_2,...,n_p},\lambda)=\lambda^{n-p}\phi(\overline{G_{n_1,n_2,...,n_p}},\lambda)$

where $n=n_1+n_2+\ldots+n_p$.

Theorem 3.10. If G is a graph on $p$ vertices with spectrum \{$\lambda^1$, $0$\}$^{p-2}$, \{-$\alpha$\}$^1$ then for any positive integers $n_1,n_2,...,n_p$:

$Spec(G_{n_1,n_2,...,n_p})=\{[\sqrt{|E^1|}]^1,[0]^{p-2},[-\sqrt{|E^1|}]^1\}$.

Proof. Now by Theorem 2.5, $Spec(G_{n_1,n_2,...,n_p})$ will be of the form \{$\theta^1$, $0$\}$^{p-2}$, \{-$\theta^1$\}.. Then from the Sachs theorem, $-\theta^2=|E^1|$ and the proof follows.

III. DETERMINANT OF QUOTIENT GRAPH

Though we can not get a similar result as in [18] for the spectral parameters of vertex multiplication graphs, we obtain a similar result for determinant of its quotient graphs.

Note 3.1. Suppose two vertices, in a graph, have same neighbors, then their corresponding rows in the adjacency matrix are equal, and consequently the determinant of the graph will be zero. Hence, instead of $G=\overline{h}$ we study the determinant of quotient graphs of $G=\overline{h}$.

Theorem 3.2. Let G be any graph on $p$ vertices and $G_{n_1,n_2,...,n_p}$ be the vertex multiplication graph $G^p$ where \( n_1, n_2, ..., n_p \). Then the determinant of the quotient graph $\overline{G_{n_1,n_2,...,n_p}}$ is $|\bigcup_{i=1}^{p}Spec(\overline{G_{n_i}})|$.

Proof. If $A(G)=[a_{ij}]$ and $A(G_{n_1,n_2,...,n_p})=[b_{ij}]$, then $b_{ij}=a_{ij}$ for all i and j. Hence $|\bigcup_{i=1}^{p}Spec(\overline{G_{n_i}})|$
Consider a graph $G$ with vertex set $\{v_1, v_2, \ldots, v_p\}$ for every vertex $v_i$ attach $k$ pendant vertices. Then the resulting graph is known as Thorn graph or Thorny graph denoted by $G^+k$ ([8], [9], and [13]).

In [17], distance $i$ graphs of cycles, paths and hadamard graphs and their corresponding spectrum have been studied. Here we continue this on study Thorn graphs $G^{+k}$ and find their distance $i$ spectrum, when $G$ is connected $r$-regular graph with diameter 2.

**Proposition 4.2.** Let $G$ be a graph with vertex set $\{v_1,v_2,\ldots,v_p\}$ and $(n_1,n_2,\ldots,n_p)$ be a $p$-tuple of positive integers. For $i \neq 2$, $T_i\left(G_{n_1,n_2,\ldots,n_p}\right) = T_i\left(G_{n_1,n_2,\ldots,n_p}\right)$ and for $i=2$, $T_2\left(G_{n_1,n_2,\ldots,n_p}\right) = T_2\left(G_{n_1,n_2,\ldots,n_p}\right)$ with additional edges, that form cliques on copies of same vertex. That is, replace each vertex $v_i$ of $T_2(G)$ by $K_{n_i}$ and all possible edges. 

**Proof.** Let $u$ and $v$ be its two vertices of $G$ and $u'$ and $v'$ be their Copies in $G_{n_1,n_2,\ldots,n_p}$. Then $d(u',v') = d(u,v)$, if $u \neq v$, and $d(u',v') = 2$, if $u = v$. Hence the proof follows from the definition of distance $i$ graph. \[\square\]

**Lemma 4.3.** [6] Let $A$, $B$, $C$ and $D$ be matrices and let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then $|M| = |AD - BC|$, if $A$ and $C$ commutes.

**Lemma 4.4.** [6] Let $G$ be a connected $r$-regular graph on $p$ vertices with an adjacency matrix $A$ having $m$ distinct eigenvalues $\lambda_1 = \epsilon_1$, $\lambda_2 = \epsilon_2$, $\ldots$, $\lambda_m$. Then there exists a polynomial $Q(x) = p^{(x-\lambda_1)(x-\lambda_2)(x-\lambda_m)}$ such that $Q(A) = I$ so that $Q(r) = p$ and $Q(A) = 0$ for all $\lambda_i \neq r$.

**Theorem 4.5.** Let $G$ be a connected $r$-regular graph with diameter 2 on $p$ vertices with spec$(G) = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p\}$. Then for $k \geq 1$, distance $i$ spectrum of $G^+k$ are given as below:

- $\text{Spec}_1(G^+k)$ consists of:
  \[\frac{1}{2}\left[\lambda_1 + \sqrt{\lambda_1^2 + 4k}\right] \text{ with multiplicity } m \text{ for } i=1 \text{ to } p; \]
  \[0 \text{ with multiplicity } p(k-1).\]

- $\text{Spec}_2(G^+k)$ consists of:
  \[\frac{1}{2}\left[k-1 + p \pm \sqrt{(k-1+p)^2 - 4k(p-kr)}\right] \text{ with multiplicity } 1; \]
  \[\frac{1}{2}\left[k-1 \pm \sqrt{(k-1)^2 + 4k}\right] \text{ with multiplicity } 1 \text{ for } i=2 \text{ to } p; \]
  \[-1 \text{ with multiplicity } p(k-1).\]

- $\text{Spec}_3(G^+k)$ consists of:
  \[\frac{1}{2}\left[kr \pm \sqrt{(kr)^2 + 4k}\right] \text{ with multiplicity } 1; \]
  \[\frac{1}{2}\left[\lambda_i \pm \sqrt{kr^2 + 4k}\right] \text{ with multiplicity } 1 \text{ for } i=2 \text{ to } p; \]
  \[0 \text{ with multiplicity } p(k-1).\]

- $\text{Spec}_4(G^+k)$ consists of:
  \[k(p-1-r) \text{ with multiplicity } 1; \]
  \[k(-1-\lambda_i) \text{ with multiplicity } 1 \text{ for } i=2 \text{ to } p; \]
  \[0 \text{ with multiplicity } pk.\]

**Proof:** First we consider the case $k = 1$, that is the graph $G^+1$. Let $\{v_1,v_2,\ldots,v_p\}$ be the vertices of $G$ and $\{v_{p+1},v_{p+2},\ldots,v_{2p}\}$ be the corresponding pendant vertices. Let $A$ and $B$ be the adjacency matrices of $G$ and $G^+1$ respectively. Adacency matrices of $T_i(G^+1)$ are:

\[
\begin{bmatrix}
I & A \\
A & I
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}
\]

for $i = 1, 2, 3, 4$ respectively. Now, $G^+k = G_{n_1,n_2,\ldots,n_p}$, where $n_i = 1$ and $n_{pi} = k$ for $j = 1, 2, \ldots, p$. Hence from Theorems 2.5, 2.7 and Proposition 4.2, spectrum of $T_i(G^+k)$ are:

\[
\begin{bmatrix}
I & A^{k-1} \\
A & I
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
0 & A
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**ACKNOWLEDGMENT**

The first author thanks National Board for Higher Mathematics, DAE, India for supporting him by NBHM Ph.D. fellowship (Grant number 2/39(2)/2009/NBHM/R&D-II/1941), since a part of this work is done by him during his Ph.D.

**REFERENCES**


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M. Saravanan has done his Ph.D. in Spectral Graph theory.

K. M. Kathiresan had done his Ph.D. in Graph theory. He has published more than 80 articles in reputed journals. He has served as reviewer in many reputed journals and he is the member of many academic societies. Also he has received “TANSA - 2012 Award (Tamilnadu Scientist Award)” awarded by TNSCST.