Global Convexity Graph of a Graph

K. Karuppasamy, S. Arumugam

Abstract: Let \( G = (V, E) \) be a graph. A function \( g : V \rightarrow [0,1] \) is called a global dominating function (GDF) of \( G \), if for every \( v \in V \), \( g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1 \) and \( g(N(v)) = \sum_{u \in N(v)} g(u) \geq 1 \). A GDF \( g \) of a graph \( G \) is minimal (MGDF) if for all functions \( f : V \rightarrow [0,1] \) such that \( f \leq g \) and \( f(v) \neq g(v) \) for at least one \( v \in V \), \( f \) is not a GDF.

In this paper, we introduce the concept of global convexity graph and determine the global convexity graphs for some standard graphs.

Keywords: convexity graph, global convexity graph, global dominating function, global domination.

I. INTRODUCTION

Let \( G = (V, E) \) be finite, undirected simple graph with \( p \) vertices and \( q \) edges. For basic terminology in graphs we refer to Chartrand and Lesniak [2].

A comprehensive treatment of the fundamentals of domination in graphs is given in the book by Haynes et al. [4]. Surveys several advanced topics on domination is given in the book edited by Haynes et al. [5].

Sampathkumar [7] introduced the concept of global domination.

A dominating set \( D \) of \( G = (V, E) \) is a global dominating set of \( G \) if \( D \) is also a dominating set of the complement \( \overline{G} \) of \( G \). The minimum cardinality of a global dominating set of \( G \) is called the global domination number of \( G \) and is denoted by \( \gamma_g(G) \) or simply \( \gamma_g \).

In [1] we have introduced the concept of fractional global domination.

Definition 1.1: A function \( g : V \rightarrow [0,1] \) is called a global dominating function (GDF) of a graph \( G = (V, E) \) if for every \( v \in V \), \( g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1 \) and \( g(N(v)) = \sum_{u \in N(v)} g(u) \geq 1 \).

A GDF \( g \) of a graph \( G \) is called minimal (MGDF) if for all functions \( f : V \rightarrow [0,1] \) such that \( f \leq g \) and \( f(v) \neq g(v) \) for at least one \( v \in V \), \( f \) is not a GDF. The fractional global domination number \( \gamma_g(G) \) and the upper fractional global domination number \( \Gamma_g(G) \) are defined as follows:

\[
\gamma_g(G) = \min \{ g | g \text{ is an MGDF of } G \} \text{ and } \\
\Gamma_g(G) = \max \{ g | g \text{ is a MGDF of } G \}.
\]

Definition 1.2: [1] Let \( f \) be a GDF of a graph \( G \). The positive set \( P_f \) of \( f \) and the boundary set \( B_f \) of \( f \) are defined as \( P_f = \{ v \in V : f(v) > 0 \} \) and \( B_f = N_f \cup \overline{N_f} \) where

\[
N_f = \{ v \in V : \sum_{u \in N[v]} f(u) = 1 \}, \quad \overline{N_f} = \{ v \in V : \sum_{u \in N(v)} f(u) = 1 \}.
\]

Definition 1.3: Let \( f \) be a GDF of a graph with positive set \( P_f \) and boundary set \( B_f \). We say that \( B_f \) globally dominates \( P_f \) if for every vertex \( v \in P_f - B_f \), there exists a vertex \( u \in N_f \) such that \( u \) is adjacent to \( v \) or there exists a vertex \( u \in \overline{N_f} \) such that \( u \) is not adjacent to \( v \) and we write \( B_f \rightarrow v \rightarrow P_f \).

Theorem 1.4: [1] A GDF \( f \) of a graph \( G \) is an MGDF if and only if \( B_f \rightarrow v \rightarrow P_f \).

Definition 1.5: Let \( f \) and \( g \) be GDFs of \( G \) and let \( 0 < \lambda < 1 \). Then \( h_f = \lambda f + (1-\lambda)g \) is called a convex combination of \( f \) and \( g \).

Theorem 1.6: [1] Let \( f \) and \( g \) be two minimal GDFs of \( G \) and let \( 0 < \lambda < 1 \). Then \( h_f = \lambda f + (1-\lambda)g \) is a minimal GDF of \( G \) if and only if \( (N_f \cap N_g) \cup (\overline{N_f} \cap \overline{N_g}) \rightarrow v \rightarrow P_f \cup P_g \).

The above theorem shows that if \( f \) and \( g \) are MGDFs of \( G \) then either all convex combinations of \( f \) and \( g \) are MGDFs or no convex combination of \( f \) and \( g \) is an MGDF. Hence analogous to the concept of convexity graph with respect to MDFs introduced by Cockayne et al. [3], we introduce the concept of global convexity graph with respect to MGDFs and initiate a study of the same. We determine the global convexity graphs of some standard graphs.

II. GLOBAL CONVEXITY GRAPH

Let \( \Omega \) denote the set of all MGDFs of a graph \( G \). We define a relation \( \rho \) on \( \Omega \) as follows. For \( f, g \in \Omega \), \( f \rho g \) if and only if \( N_f = N_g, \overline{N_f} = \overline{N_g} \) and \( P_f = P_g \). Clearly \( \rho \) is an equivalence relation on \( \Omega \) and \( \Omega \) is partitioned into a finite number of equivalence classes.

Definition 2.1: Let \( X = \{X_1, X_2, \ldots, X_s\} \) be the set of all equivalence classes of \( \Omega \) with respect to the equivalence relation \( \rho \). We define the global convexity graph (GCG) of \( G \), denoted by \( C_s(G) \), by \( V(C_s(G)) = X = \{X_1, X_2, \ldots, X_s\} \) and \( X_i, X_j \) are adjacent if and only if there exist \( f \in X_i \) and \( g \in X_j \) such that any convex combination of \( f \) and \( g \) is an MGDF of \( G \).
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We proceed to determine the global convexity graphs of some standard graphs. The following observation is useful in this regard.

Observation 2.2: If there exists a dominating set $S$ of $G$ such that $N_f(S)$ contains $S$ for all MGDFs $f$ of $G$ or if there exists a dominating set $T$ of $G$ such that $N_f(T)$ contains $T$ for all MGDFs $f$ of $G$, then it follows from Theorem 1.6 that $C_g(G)$ is complete.

Observation 2.3: For any complete graph $K_n$, we have $C_g(K_n) \equiv K_1$. If $g(v) = 1$ for all $v \in V(K_n)$.

Theorem 2.4: For the complete bipartite graph $G = K_{r,s}$, we have $C_g(G) \equiv K_{[r-1][s-1]}$.

Proof: Let $(X,Y)$ be the bipartition of $G$ with $|X| = r$ and $|Y| = s$. Let $S$ and $T$ be nonempty subsets of $X$ and $Y$ respectively.

Define $g : V(G) \to [0,1]$ by $g(u) = 1$ if $u \in S$; $g(u) = 1/|T|$ if $u \in T$; $g(u) = 0$ otherwise.

Then $g$ is an MGDF of $G$ with $P_g = S \cup T$, $N_g = V(G) - (S \cup T)$ and $N_f = X \cup Y = V(G)$. Thus each pair $(S,T)$ where $S$ is a non-empty subset of $X$ and $T$ is a non-empty subset of $Y$ gives an MGDF of $G$ and if $(S_1,T_1) = (S_2,T_2)$, then the corresponding MGDFs determine two distinct equivalence classes. Thus we have $2^{[r-1]}2^{[s-1]}$ equivalence classes of MGDFs.

We now claim that these are the only equivalence classes of MGDFs for $G$. Let $f$ be any MGDF of $G$ with $P_f = S' \cup T'$ where $S' \subseteq X$ and $T' \subseteq Y$. Clearly if $u \in X$, then

$$\sum_{v \in N_f(u)} f(v) = \sum_{v \in S} f(v) \geq 1.$$ Similarly, $\sum_{v \in T} f(v) \geq 1$. It follows from the minimality that $\sum_{v \in S} f(v) = 1$ and $\sum_{v \in T} f(v) = 1$. Hence the number of equivalence classes of MGDFs of $G$ is $2^{[r-1]}2^{[s-1]}$. Also $N_f = V$ for all MGDFs and it follows from Observation 2.2 that $C_g(G)$ is complete. Thus $C_g(G) \equiv K_{[r-1][s-1]}$.

Corollary 2.5: For the star $K_{1,n}$, we have $C_g(K_{1,n}) \equiv K_{[n-1][n-1]}$.

Theorem 2.6: For the wheel $W_5 = K_1 + C_4$, we have $C_g(W_5) \equiv K_9$.

Proof: Let $V(W_5) = \{v_0,v_1,v_2,v_3,v_4\}$ and $E(W_5) = \{v_0v_i : 1 \leq i \leq 4\} \cup \{v_1v_2,v_1v_3,v_2v_3,v_4v_1\}$. Let $g$ be any MGDF of $W_5$. Since $v_0$ is an isolated vertex in $W_5$, $g(v_0) = 1$. Also

$$\sum_{v \in N_f(v_0)} g(v) = g(v_1) + g(v_3) \geq 1$$ and

$$\sum_{v \in N_f(v_1)} g(u) = g(v_2) + g(v_4) \geq 1.$$ It follows from the minimality of $g$ that $g(v_1) + g(v_3) \geq 1$ and $g(v_2) + g(v_4) \geq 1$.

Now, let $g(v_1) = 1$, $g(v_3) = 1$, $g(v_2) = 0$, $g(v_4) = 0$, and $g(v) = 1$ except for $u = v_4$, where $0 \leq 1 \leq 0$. Hence there are four equivalence classes of MGDFs of $W_5$. Since $g(v_1) + g(v_3) = 1$ and $g(v_2) + g(v_4) = 0$ it follows that $N_f = V$ for all MGDF $g$. Hence it follows from Observation 2.2 that $C_g(W_5)$ is complete. Thus $C_g(W_5) \equiv K_9$.

Theorem 2.7: For any graph $G$ on $n$ vertices with $\Delta(G) < n - 1$, we have $C_g(G \setminus K_n) \equiv K_n$.

Proof: Let $V(G) = \{v_1,v_2,\ldots,v_n\}$ and let $u_1,u_2,\ldots,u_k$ be the pendant vertices of $G \setminus K_n$ adjacent to $v_1,v_2,\ldots,v_k$, respectively. Define $g : V(G \setminus K_n) \to [0,1]$ by $g(v_1) = 1$ and $g(u_i) = 0$ for $i = 1,2,\ldots,k$. We claim that $g$ is an MGDF of $G \setminus K_n$. Obviously, $\sum_{v \in N_f(v_1)} g(v) = 1$ and $\sum_{v \in N_f(v_i)} g(v) = 1$ for all $i = 1,2,\ldots,n$.

Also $\sum_{v \in N_f(v_i)} g(v) = 1$. Since $\Delta(G) < n - 1$, there exists a vertex $v_i$ such that $v_i$ is not adjacent to $v_1$ and hence $\sum_{v \in N_f(v_i)} g(v) \geq g(v_1) + g(v_i) + g(u_i) \geq 1$. Hence $g$ is a DGF of $G \setminus K_n$.

Since $g(u_i) + g(v_1) = 1$ for all $i$, it follows that $g$ is an MGDF of $G \setminus K_n$. If $v_i = 0$, then $u_i \in P_f$ and $u_i \notin P_f$. If $v_i = 1$, then $v_i \in P_f$ and $u_i \notin P_f$. If $v_i < 1$, then $v_i \in P_f$. Hence there exist three MGDFs of $G \setminus K_n$.

Thus we have 3$^n$ MGDFs and all of them have distinct positive sets. Thus we have 3$^n$ equivalence classes of MGDFs of $G \setminus K_n$. We now claim that these are the only equivalence classes of MGDFs for $G \setminus K_n$.

Let $f$ be any MGDF of $G \setminus K_n$.

Then $\sum_{v \in N_f(v_i)} g(v) = f(u_i) + f(v_1) \geq 1$. It follows from the minimality of $f$ that $f(u_i) + f(v_1) \geq 1$ for all $i$. Hence $V(C_g(G \setminus K_n)) = 3^n$. Also for all MGDFs $g$ of $G \setminus K_n$, $N_g$ contains the dominating set of $\{u_1,u_2,\ldots,u_k\}$.

Hence it follows from Observation 2.2 that the convexity graph of $G \setminus K_n$ is complete. Thus $C_g(G \setminus K_n) \equiv K_n$.

It follows from Corollary 2.5 that the convexity graph of the path $P_2$ is isomorphic to $K_4$. We now proceed to find the convexity graphs of the paths $P_3$ and $P_4$.

Theorem 2.8: For the path $P_3$, we have $C_g(P_3) \equiv K_{3}$.

Proof: Let $P_3 = (v_1,v_2,v_3)$ and $g = (\lambda_1,\lambda_2,\lambda_3)$ be any MGDF of $P_3$ where $g(v_1) = \lambda_2 \leq \lambda_3$. Then $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_1 + \lambda_2 + \lambda_3 \geq 1$ and $\lambda_2 + \lambda_3 \geq 1$. Hence it follows from Observation 2.2 that the global convexity graph of $P_3$ is complete. Hence $C_g(P_3) \equiv K_{3}$.

Theorem 2.9: For the path $P_4$, we have $C_g(P_4) \equiv K_{24}$.

Proof: Let $P_4 = (v_1,v_2,v_3,v_4)$ and $g = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ be any MGDF of $P_4$ where $g(v_1) = \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_1$. Clearly at least one of the vertices $v_1$ or $v_4$ and at least one of the vertices $v_2$ or $v_3$ are in $P_4$. Let $S = \{v_1,v_2,v_3,v_4\}$. 

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If \( P_3 \cap S = \{v_1, v_3\} \), then \( \lambda_2 = \lambda_4 = 0 \) and hence we have exactly one equivalence class, namely, \( g_1 = (0,1,1,0) \) with \( P_{g_1} = \{v_1, v_3, v_5\} \), \( N_{g_1} = \{v_1, v_3, v_5\} \) and \( \tilde{N}_{g_1} = \{v_1, v_3\} \).

If \( P_3 \cap S = \{v_1, v_2\} \), then \( \lambda_2 = \lambda_4 = 0 \) and hence we have exactly one equivalence class, namely, \( g_2 = (1,0,1,0) \) with \( P_{g_2} = \{v_1, v_2, v_3\} \), \( N_{g_2} = \{v_1, v_2, v_3\} \) and \( \tilde{N}_{g_2} = \{v_1, v_2, v_3\} \).

Similarly, if \( P_3 \cap S = \{v_2, v_3\} \), then we have exactly one equivalence class, namely, \( g_3 = (0,1,0,1) \) with \( P_{g_3} = \{v_2, v_3, v_5\} \), \( N_{g_3} = \{v_1, v_3\} \) and \( \tilde{N}_{g_3} = \{v_2, v_3\} \).

Now suppose \( P_3 \cap S = \{v_1, v_4\} \). Then \( \lambda_2 = 0 \). Since

\[
\sum_{uv \not\in (v_1)} g(u) = g(v_1),
\]

it follows \( g(v_1) = 1 \) and we have exactly

one equivalence class, namely, \( g_4 = (1,1,1,0) \) with \( P_{g_4} = \{v_2, v_3, v_4\} \), \( N_{g_4} = \{v_1, v_3\} \) and \( \tilde{N}_{g_4} = \{v_3\} \).

If \( P_3 \cap S = \{v_1, v_2\} \), then we have exactly one equivalence class, namely, \( g_5 = (1,1,0,0) \) where \( 0 < \lambda_2 \leq 1 \) with \( P_{g_5} = V - \{v_2\}, N_{g_5} = \{v_1, v_3, v_5\} \) and \( \tilde{N}_{g_5} = \{v_1, v_3\} \).

Similarly, if \( P_3 \cap S = \{v_1, v_3\} \), then we have exactly one equivalence class, namely, \( g_6 = (0,1,1,1) \) where \( 0 < \lambda_2 < 1 \) with \( P_{g_6} = V - \{v_2\}, N_{g_6} = \{v_1, v_3\} \) and \( \tilde{N}_{g_6} = \{v_1, v_3\} \).

If \( P_3 \cap S = \{v_2, v_3\} \), then we get exactly one equivalence class, namely, \( g_7 = (0,1,0,1) \) where \( 0 < \lambda_2 < 1 \) with \( P_{g_7} = V - \{v_2\}, N_{g_7} = \{v_1, v_3\} \) and \( \tilde{N}_{g_7} = \{v_2, v_3\} \).

If \( P_3 \cap S = \{v_2, v_4, v_5\} \). Then \( 0 < \lambda_2 < 1 \), \( \lambda_1 = 1 - \lambda_2 \), \( 0 < \lambda_4 < 1 \), \( \lambda_4 = 1 - \lambda_2 \), we claim that in this case, there exist three equivalence classes of MGDFs.

If \( \lambda_3 = 0 \), then

\[
\sum_{uv \not\in (v_1)} g(u) \geq 1 \quad \text{implies that} \quad \lambda_1 + \lambda_4 \geq 1, \quad \text{so that} \quad \lambda_2 \geq 1 - \lambda_4.
\]

Since \( g \) is an MGDF, \( \lambda_1 = 1 - \lambda_4 \). Hence we get an equivalence class \( g_8 = (\lambda_1, 1 - \lambda_1, 0, 1 - \lambda_1) \) where \( 0 < \lambda_1 < 1 \) with \( P_{g_8} = V - \{v_1\}, N_{g_8} = V \) and \( \tilde{N}_{g_8} = \{v_1\} \).

If \( \lambda_3 > 0 \), then

\[
\sum_{uv \not\in (v_1)} g(u) \geq 1 \quad \text{gives} \quad \lambda_1 \geq 1 - \lambda_4 - \lambda_1 = \lambda_4 - 1. \quad \text{Also} \quad \sum_{uv \not\in (v_1)} g(u) \geq 1 \quad \text{gives} \quad \lambda_1 \geq 1 - \lambda_4 - \lambda_1 = \lambda_4 + 1. \quad \text{Since} \quad g \quad \text{is an MGDF of} \quad G \quad \text{it follows} \quad \lambda_1 = \max(1 - \lambda_4 - \lambda_1, \lambda_4 + 1). \quad \text{Also since} \quad \lambda_3 > 0, \quad \text{it follows that} \quad \lambda_1 + \lambda_4 = 1. \quad \text{Hence we have two equivalence classes of} \quad \text{MGDFs according as} \quad \lambda_1 + \lambda_4 < 1 \quad \text{or} \quad \lambda_1 + \lambda_4 > 1 \text{and they are given by} \quad g_{10} = (\lambda_1, 1 - \lambda_1, 0, 1 - \lambda_1) \quad \text{where} \quad 0 < \lambda_1, \lambda_4 < 1 \quad \text{and} \quad \lambda_4 + 1 < 1 \quad \text{with} \quad P_{g_{10}} = V, N_{g_{10}} = \{v_1, v_3\} \quad \text{and} \quad \tilde{N}_{g_{10}} = \{v_3\} \quad \text{and} \quad g_{11} = (\lambda_1, 1 - \lambda_1, \lambda_4 + 1, -1 - \lambda_1, - \lambda_1) \quad \text{where}
\]

Thus there are exactly 11 equivalence classes of MGDFs for \( P_3 \).

Let \( A = \{g_1, g_3, g_5, g_7\} \), \( B = \{g_2, g_4, g_6, g_8\} \) and \( C = \{g_9, g_{10}, g_{11}\} \).

By using Theorem 1.6, it can be easily verified that \( A \cup B \cup C \equiv (K_2 + K_1) \) and \( (A \cup B) \equiv 2K_1 \). Also every element of \( A \cup B \cup C \) is adjacent to every element of \( C \). Hence \( A \cup B \cup C \equiv (2K_1 + K_1) \) and thus \( C_6 (P_3) \equiv (2K_4 + K_1) \).

III. Conclusion

In this paper we have introduced the concept of global convexity graphs and determined the global convexity graphs of some standard graphs. An interesting problem is to investigate properties of the global convexity graphs of specific families of graphs such as trees and cycles. Also the problem of determining the global convexity graphs of arbitrary paths and cycles remains open.

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