International Journal of Engineering and Advanced Technology (IJEAT)
ISSN: 2249 – 8958, Volume-9, Issue-1S4, December 2019

Global Convexity Graph of a Graph

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***Abstract:** Let G = (V, E) be a graph. A function $g: V \to [0,1]$ is called a global dominating function (GDF) of G, if for every $v \in V, g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ and $g(\overline{N(v)}) = \sum_{u \notin N(v)} g(u) \geq 1$. A GDF g of a Graph G is minimal (MGDF) if for all functions $f: V \to [0,1]$ such that $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V$, f is not a GDF. In this paper, we introduce the concept of global convexity graph and determine the global convexity graphs for some standard graphs.*

***Keywords:** convexity graph, global convexity graph, global dominating function, global domination.*

I. INTRODUCTION

Let $G = (V, E)$ be finite, undirected simple graph with p vertices and q edges. For basic terminology in graphs we refer to Chartrand and Lesniak [2].

A comprehensive treatment of the fundamentals of domination in graphs is given in the book by Haynes et al. [4]. Surveys several advanced topics on domination is given in the book edited by Haynes et al. [5].

Sampathkumar [7] introduced the concept of global domination.

A dominating set D of $G = (V, E)$ is a global dominating set of G if D is also a dominating set of the complement $\overline{G}$ of G. The minimum cardinality of a global dominating set of G is called the global domination number of G and is denoted by $\gamma_g(G)$ or simply γ_g.

In [1] we have introduced the concept of fractional global domination.

Definition 1.1:[1] A function $g: V \to [0,1]$ is called a global dominating function (GDF) of a graph $G=(V,E)$ if for every $v \in V, g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ and $g(\overline{N(v)}) = \sum_{u \notin N(v)} g(u) \geq 1$.

A GDF g of a graph G is called minimal (MGDF) if for all functions $f: V \to [0,1]$ such that $f \leq g$ and $f(v) \neq g(v)$ for at least one $v \in V$, f is not a GDF. The fractional global domination number $\gamma_{fg}(G)$ and the upper fractional global domination number $\Gamma_{fg}(G)$ are defined as follows:

$\gamma_{fg}(G) = \min\{|g| : g \text{ is an MGDF of } G\}$ and

$\Gamma_{fg}(G) = \max\{|g| : g \text{ is a MGDF of } G\}$.

Definition 1.2: [1] Let f be a GDF of a graph G. The positive set P_f of f and the boundary set B_f of f are defined as $P_f = \{v \in V : f(v) > 0\}$ and $B_f = N_f \cup \overline{N_f}$ where $N_f = \left\{v \in V : \sum_{w \in N[v]} f(w) = 1\right\}$, $\overline{N_f} = \left\{v \in V : \sum_{w \notin N(v)} f(w) = 1\right\}$.

Definition 1.3: Let f be a GDF of a graph with positive set P_f and boundary set B_f. We say that B_f globally dominates P_f if for every vertex $v \in P_f - B_f$, there exists a vertex $u \in N_f$ such that u is adjacent to v or there exists a vertex $u \in \overline{N_f}$ such that u is not adjacent to v and we write $B_f \xrightarrow[g]{} P_f$.

Theorem 1.4: [1] A GDF f of a graph G is an MGDF if and only if $B_f \xrightarrow[g]{} P_f$.

Definition 1.5: Let f and g be GDFs of G and let $0 < \lambda < 1$. Then $h_\lambda = \lambda f + (1-\lambda)g$ is called a convex combination of f and g.

Theorem 1.6: [1] Let f and g be two minimal GDFs of G and let $0 < \lambda < 1$. Then $h_\lambda = \lambda f + (1-\lambda)g$ is a minimal GDF of G if and only if $(N_f \cap N_g) \cup (\overline{N_f} \cap \overline{N_g}) \xrightarrow[g]{} P_f \cup P_g$.

The above theorem shows that if f and g are MGDFs of G then either all convex combinations of f and g are MGDFs or no convex combination of f and g is an MGDF. Hence analogous to the concept of convexity graph with respect to MDFs introduced by Cockayne et al. [3], we introduce the concept of global convexity graph with respect to MGDFs and initiate a study of the same. We determine the global convexity graphs of some standard graphs.

II. GLOBAL CONVEXITY GRAPH

Let Ω denote the set of all MGDFs of a graph G. We define a relation ρ on Ω as follows. For $f, g \in \Omega$, $f \rho g$ if and only if $N_f = N_g, \overline{N_f} = \overline{N_g}$ and $P_f = P_g$. Clearly ρ is an equivalence relation on Ω and Ω is partitioned into a finite number of equivalence classes.

Definition 2.1: Let $X = \{X_1, X_2, \cdots, X_t\}$ be the set of all equivalence classes of Ω with respect to the equivalence relation ρ. We define the global convexity graph (GCG) of relation ρ. We define the global convexity graph (GCG) of G, denoted by $C_g(G)$, by $V(C_g(G)) = X = \{X_1, X_2, \cdots, X_t\}$ and X_i, X_j are adjacent if and only if there exist $f \in X_i$ and $g \in X_j$ such that any convex combination of f and g is an MGDF of G.

Revised Manuscript Received on December 15, 2019.

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We proceed to determine the global convexity graphs of some standard graphs. The following observation is useful in this regard.

Observation 2.2: If there exists a dominating set S of G such that N_f contains S for all MGDFs f of G or if there exists a dominating set T of $\overline{G}$ such that $\overline{N_f}$ contains T for all MGDFs f of G, then it follows from Theorem 1.6 that $C_g(G)$ is complete.

Observation 2.3: For any complete graph K_n, we have $C_g(K_n) \cong K_1$. In fact if g is any MGDF of K_n, then $g(v)=1$ for all $v \in V(K_n)$.

Theorem 2.4: For the complete bipartite graph $G = K_{r,s}$, we have $C_g(G) \cong K_{(2^r-1)(2^s-1)}$.

Proof: Let (X,Y) be the bipartition of G with $|X| = r$ and $|Y| = s.$ Let S and T be nonempty subjects of X and Y respectively.

Define $g: V(G) \to [0,1]$ by $g(u) = \frac{1}{|S|}$ if $u \in S$; $g(u) = \frac{1}{|T|}$ if $u \in T; g(u) = 0$ otherwise

Then g is an MGDF of G with $P_g = S \cup T$, $N_g = V(G) - (S \cup T)$ and $\overline{N_g} = X \cup Y = V(G)$. Thus each pair (S,T) where S is a nonempty subset of X and T is a nonempty subset of Y gives an MGDF of G and if $(S_1,T_1) \neq (S_2,T_2)$, then the corresponding MGDFs determine two distinct equivalence classes. Thus we have $(2^r-1)(2^s-1)$ equivalence classes of MGDFs.

We now claim that these are the only equivalence classes of MGDFs for G. Let f be any MGDF of G with $P_f = S' \cup T'$ where $S' \subseteq X$ and $T' \subseteq Y$. Clearly if $u \in X$, then $\sum_{v \notin N(u)} f(v) = \sum_{v \in S'} f(v) \geq 1$. Similarly, $\sum_{v \in T'} f(v) \geq 1$. It follows from the minimality that $\sum_{v \in S'} f(v) = 1$ and $\sum_{v \in T'} f(v) = 1$. Hence the number of equivalence classes of MGDFs of G is $(2^r-1)(2^s-1)$. Also $\overline{N_f} = V$ for all MGDFs and it follows from Observation 2.2 that $C_g(G)$ is complete. Thus $C_g(G) \cong K_{(2^r-1)(2^s-1)}$.

Corollary 2.5: For the star $K_{1,n}$, we have $C_g(K_{1,n}) \cong K_{2^n-1}$.

Theorem 2.6: For the wheel $W_5 = K_1 + C_4$, we have $C_g(W_5) \cong K_9$.

Proof: Let $V(W_5) = \{v_0, v_1, v_2, v_3, v_4\}$ and $E(W_5) = \{v_0 v_i : 1 \leq i \leq 4\} \cup \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let g be any MGDF of W_5. Since v_0 is an isolated vertex in $\overline{W_5}$, $g(v_0) = 1$. Also $\sum_{u \notin N(v_1)} g(u) = g(v_1) + g(v_3) \geq 1$ and $\sum_{u \notin N(v_2)} g(u) = g(v_2) + g(v_4) \geq 1$. It follows from the minimality of g that $g(v_1) + g(v_3) = 1$ and $g(v_2) + g(v_4) = 1$.

Now, let $g(v_1) = \lambda_1$ and $g(v_2) = \lambda_2$ so that $g(v_3) = 1 - \lambda_1$ and $g(v_4) = 1 - \lambda_2$, where $0 \leq \lambda_1, \lambda_2 \leq 1$. Hence there are nine equivalence classes of MGDFs of W_5. Since $g(v_1) + g(v_3) = g(v_2) + g(v_4) = g(v_0) = 1$ it follows that $\overline{N_g} = V$ for all MGDF g. Hence it follows from Observation 2.2 that $C_g(W_5)$ is complete. Thus $C_g(W_5) \cong K_9$.

Theorem 2.7: For any graph G on n vertices with $\Delta(G) < n-1$, we have $C_g(G \circ K_1) \cong K_{3^n}$.

Proof: Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and let $u_1, u_2, \cdots, u_n$ be the pendent vertices of $G \circ K_1$ adjacent to $v_1, v_2, \cdots, v_n$ respectively. Define $g: V(G \circ K_1) \to [0,1]$ by $g(v_i) = \lambda_i$ and $g(u_i) = 1 - \lambda_i$ where $0 \leq \lambda_i \leq 1$. We claim that g is an MGDF of $G \circ K_1$. Obviously, $\sum_{w \in N[u_i]} g(w) = 1$ and $\sum_{w \in N[v_i]} g(w) \geq 1$ for all $i = 1, 2, \cdots, n$. Also $\sum_{w \notin N(u_i)} g(w) \geq 1$. Since $\deg(v_i) \leq \Delta(G) < n-1$, there exists a vertex v_k such that v_k is not adjacent to v_i and hence $\sum_{w \notin N(v_i)} g(w) \geq g(v_i) + g(v_k) + g(u_k) \geq 1$. Hence g is a GDF of $G \circ K_1$. Since $g(u_i) + g(v_i) = 1$ for all i, it follows that g is an MGDF of $G \circ K_1$. If $\lambda_i = 0$, then $u_i \in P_f$ and $v_i \notin P_f$. If $\lambda_i = 1$, then $v_i \in P_f$ and $u_i \notin P_f$. If $0 < \lambda_i < 1$, then $u_i, v_i \in P_f$. Thus we have 3^n MGDFs and all of them have distinct positive sets. Thus we have 3^n equivalence classes of MGDFs of $G \circ K_1$. We now claim that these are the only equivalence classes of MGDFs for $G \circ K_1$.

Let f be any MGDF of $G \circ K_1$. Then $\sum_{w \in N[u_i]} f(w) = f(u_i) + f(v_i) \geq 1$. It follows from the minimality of f that $f(u_i) + f(v_i) = 1$ for all i. Hence $|V(C_g(G \circ K_1))| = 3^n$. Also for all MGDFs g of $G \circ K_1$, N_g contains the dominating set $\{u_1, u_2, \cdots, u_n\}$ of $G \circ K_1$. Hence it follows from Observation 2.2 that the convexity graph of $G \circ K_1$ is complete. Thus $C_g(G \circ K_1) \cong K_{3^n}$.

It follows from Corollary 2.5 that the convexity graph of the path P_3 is isomorphic to K_3. We now proceed to find the convexity graphs of the paths P_4 and P_5.

Theorem 2.8: For the path P_4, we have $C_g(P_4) \cong K_3$.

Proof: Let $P_4 = (v_1, v_2, v_3, v_4)$ and let $g = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be any MGDF of P_4 where $g(v_i) = \lambda_i, 0 \leq \lambda_i \leq 1$. Then $\lambda_1 + \lambda_2 = 1, \lambda_3 + \lambda_4 = 1, \lambda_1 + \lambda_3 \geq 1$ and $\lambda_2 + \lambda_4 \geq 1$. Hence it follows that $\lambda_2 = \lambda_3 = \lambda$, so that $g = (1-\lambda, \lambda, \lambda, 1-\lambda)$ where $0 \leq \lambda \leq 1$. Hence there exist exactly three equivalence classes of MGDFs of P_4, corresponding to $\lambda = 1, \lambda = 0$ or $0 < \lambda < 1$ and for each of these MGDFs, N_g contains the dominating set $\{v_1, v_4\}$ of P_4. Hence it follows from observation 2.2 that the global convexity graph of P_4 is complete. Hence $C_g(P_4) \cong K_3$.

Theorem 2.9: For the path P_5, we have $C_g(P_5) \cong 2K_4 + K_3$.

Proof: Let $P_5 = (v_1, v_2, v_3, v_4, v_5)$ and $g = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be any MGDF of P_5 where $g(v_i) = \lambda_i, 0 \leq \lambda_i \leq 1$. Clearly at least one of the vertices v_1 or v_2 and at least one of the vertices v_4 or v_5 are in P_g. Let $S = \{v_1, v_2, v_4, v_5\}$.

Retrieval Number: A12031291S419/2019©BEIESP
DOI: 10.35940/ijeat.A1203.1291S419

Published By:
Blue Eyes Intelligence Engineering
& Sciences Publication

If $P_g \cap S = \{v_1, v_5\}$, then $\lambda_2 = \lambda_4 = 0$ and hence we have exactly one equivalence class, namely, $g_1 = (1,0,1,0,1)$ with $P_{g_1} = \{v_1, v_3, v_5\}$, $N_{g_1} = \{v_1, v_3, v_5\}$ and $\overline{N_{g_1}} = \{v_2, v_4\}$.

If $P_g \cap S = \{v_1, v_4\}$, then $\lambda_5 = \lambda_2 = 0$ and hence we have exactly one equivalence class, namely, $g_2 = (1,0,0,1,0)$ with $P_{g_2} = \{v_1, v_4\}$, $N_{g_2} = V$ and $\overline{N_{g_2}} = \{v_2, v_3, v_5\}$.

Similarly, if $P_g \cap S = \{v_2, v_5\}$, then have exactly one equivalence class, namely, $g_3 = (0,1,0,0,1)$ with $P_{g_3} = \{v_2, v_5\}$, $N_{g_3} = V$ and $\overline{N_{g_3}} = \{v_1, v_3, v_4\}$.

Now suppose $P_g \cap S = \{v_2, v_4\}$. Then $\lambda_1 = \lambda_5 = 0$. Since $\sum_{u \notin N(v_3)} g(u) = g(v_3)$, it follows $g(v_3) = 1$ and we have exactly one equivalence class, namely, $g_4 = (0,1,1,1,0)$ with $P_{g_4} = \{v_2, v_3, v_4\}$, $N_{g_4} = \{v_1, v_5\}$ and $\overline{N_{g_4}} = \{v_3\}$.

If $P_g \cap S = \{v_1, v_2, v_5\}$ $g(v_4) = 0$ and hence $g(v_5) = 1$. Now let $g(v_1) = \lambda$ where $0 < \lambda < 1$. Then $g(v_2) = 1 - \lambda$. Now, $\sum_{u \in N[v_3]} g(u) \geq 1$ gives $g(v_3) = \lambda_3 > \lambda$ and since g is an MGDF, it follows $\lambda_3 = \lambda$. Hence have exactly one equivalence class, namely, $g_5 = (\lambda, 1-\lambda, \lambda, 0, 1)$ where $0 < \lambda < 1$ with $P_{g_5} = V - \{v_4\}$, $N_{g_5} = \{v_1, v_3, v_5\}$ and $\overline{N_{g_5}} = \{v_4\}$.

Similarly, if $P_g \cap S = \{v_1, v_2, v_4\}$, then we have exactly one equivalence class, namely, $g_6 = (1-\lambda, \lambda, \lambda, 1, 0)$ where $0 < \lambda < 1$ with $P_{g_6} = V - \{v_5\}$, $N_{g_6} = \{v_1, v_5\}$ and $\overline{N_{g_6}} = \{v_3\}$.

If $P_g \cap S = \{v_1, v_4, v_5\}$, then we get exactly one equivalence class, namely, $g_7 = (1, 0, \lambda, 1-\lambda, \lambda)$ where $0 < \lambda < 1$ with $P_{g_7} = V - \{v_2\}$, $N_{g_7} = \{v_1, v_3, v_5\}$ and $\overline{N_{g_7}} = \{v_2\}$.

If $P_g \cap S = \{v_2, v_4, v_5\}$, then we get exactly one equivalence class, namely, $g_8 = (0, 1, \lambda, \lambda, 1-\lambda)$ where $0 < \lambda < 1$ with $P_{g_8} = V - \{v_1\}$, $N_{g_8} = \{v_1, v_5\}$ and $\overline{N_{g_8}} = \{v_3\}$.

Now let $P_g \cap S = \{v_1, v_2, v_4, v_5\} = S$. Then $0 < \lambda_1 < 1$, $\lambda_2 = 1 - \lambda_1$, $0 < \lambda_5 < 1$, $\lambda_4 = 1 - \lambda_5$. we claim that in this case, there exist three equivalence classes of MGDFs.

If $\lambda_3 = 0$, then $\sum_{u \notin N(v_3)} g(u) \geq 1$ implies that $\lambda_1 + \lambda_5 \geq 1$, so that $\lambda_5 \geq 1 - \lambda_1$. Since g is an MGDF, $\lambda_5 = 1 - \lambda_1$. Hence we get an equivalence class $g_9 = (\lambda_1, 1-\lambda_1, 0, \lambda_1, 1-\lambda_1)$ where $0 < \lambda_1 < 1$ with $P_{g_9} = V - \{v_3\}$, $N_{g_9} = V$ and $\overline{N_{g_9}} = \{v_3\}$.

Now suppose $\lambda_3 > 0$. Then $\sum_{u \notin N(v_3)} g(u) \geq 1$ gives $\lambda_3 \geq 1 - \lambda_1 - \lambda_5$. Also $\sum_{u \in N[v_3]} g(u) \geq 1$ gives $\lambda_3 \geq 1 - \lambda_2 - \lambda_4 = \lambda_1 + \lambda_5 - 1$. Since g is an MGDF of G it follows $\lambda_3 = \max\{1 - \lambda_1 - \lambda_5, \lambda_1 + \lambda_5 - 1\}$. Also since $\lambda_3 > 0$, it follows that $\lambda_1 + \lambda_5 \neq 1$. Hence we have two equivalence classes of MGDFs according as $\lambda_1 + \lambda_5 < 1$ or $\lambda_1 + \lambda_5 > 1$ and they are given by $g_{10} = (\lambda_1, 1-\lambda_1, 1-\lambda_1-\lambda_5, 1-\lambda_5, \lambda_5)$ where $0 < \lambda_1, \lambda_5 < 1$ and $0 < \lambda_1 + \lambda_5 < 1$ with $P_{g_{10}} = V$, $N_{g_{10}} = \{v_1, v_5\}$ and $\overline{N_{g_{10}}} = \{v_3\}$ and $g_{11} = (\lambda_1, 1-\lambda_1, \lambda_1 + \lambda_5 - 1, 1-\lambda_5, \lambda_5)$ where $0 < \lambda_1, \lambda_5 < 1$ and $\lambda_1 + \lambda_5 > 1$ with $P_{g_{11}} = V$, $N_{g_{11}} = \{v_1, v_3, v_5\}$ and $\overline{N_{g_{11}}} = \phi$.

Thus there are exactly 11 equivalence classes of MGDFs for P_5. Let $A = \{g_4, g_6, g_8, g_{10}\}$, $B = \{g_1, g_5, g_7, g_{11}\}$ and $C = \{g_2, g_3, g_9\}$.

By using Theorem 1.6, it can be easily verified that $\langle A \rangle = \langle B \rangle \cong K_4, \langle C \rangle = K_3$ and $\langle A \cup B \rangle = 2K_4$. Also every element of $A \cup B$ is adjacent to every element of C. Hence $\langle A \cup B \cup C \rangle \cong (2K_4 + K_3)$ and thus $C_g(P_5) \cong (2K_4 + K_3)$.

III. CONCLUSION

In this paper we have introduced the concept of global convexity graphs and determined the global convexity graphs of some standard graphs. An interesting problem is to investigate properties of the global convexity graphs of specific families of graphs such as trees and cycles. Also the problem of determining the global convexity graphs of arbitrary paths and cycles remains open.

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