International Journal of Engineering and Advanced Technology (IJEAT)
ISSN: 2249-8958 (Online), Volume-9 Issue-1S4, December 2019

A Note on I-Sequentially Compact Spaces

Prakash B., Ananda Priya B.

Abstract—this paper investigates the properties of convergence and sequentially compact for the generalized concepts –I-convergent and I -sequentially compact. We extend Cartesian product properties of convergence to I-convergence. Further, we develop some properties of -sequential compactness of a topologicalspace.

Keywords: Ideal,I-convergentsequence,I-clusterpoint, I –sequentially compactspace.

I. INTRODUCTION

Convergent sequence in real numbers has been extended to statistical convergent separately by two different authors Fast[6] and Schoenberg [10]. Convergent sequenceimplies statisticallyconvergent.Butastatisticallyconvergentsequence need not be a convergent sequence. In general, statistically convergent sequences hold most of the properties of ordinary convergent sequences in metricspaces.

I-convergence of real sequences [7], [8] is a generalized concept of statistical convergence which is totally based on the structure of the ideal of subsets of the set of all natural numbers. In the recent literature, several workson I- convergenceincludingremarkablecontributionsbyŠalátet al. have occurred [2], [5], [7], [8], [3], [9]. I-convergence will become ordinary convergence if $I = \{I \subset N: n(I) < \infty\}$ anditwillbecomestatistical convergenceif $I = \{I \subset N: d(I) = 0\}$ where $d(A) = \lim_{n\to\infty} \frac{|\{m\in A: m\le n\}|}{n}$

We recapitulate the following definitions.

IfXisanon-emptyset,thenafamilyofsets$\mathbf{I}\subset 2^X$iscalled an ***ideal***if

1) $I_1, I_2 \in \mathbf{I}$implies$I_1 \cup I_2 \in \mathbf{I}$and
2) $I_1 \in \mathrm{I}$, $I_2 \subset I_1$ implies $I_2 \in \mathrm{I}$.

Theideal I is called ***nontrivial***if I is singletonsethaving ∅ as an element and X is not an element of I.Anon-trivial ideal I is called ***admissible***if it containsallfinitesubsets of X. Several examples of non-trivialadmissibleidealisgivenin[8].Throughoutthispaper,weconsider I asanon-trivial ideal of ℕ, the set of all natural numbers and (X, τ) denotes the topological space.

Also $\{x_n\}$denote a sequence of elements of a topological space X. An element $y \in X$ is called an I-cluster point of $\{x_n\}$ if $\{n \in \mathbb{N}: x_n \in U\} \notin I$, for every open set U containing y.

II. I-CONVERGENT SEQUENCE

Definition II.1. [4] Let (X, τ) be a topological space. A sequence $\{x_n\}$is said to be I-*convergent* to an element $x_0 \in X$if for every non-empty open neighborhood U containing x_0,$\{n \in N: x_n{}^{-1} \notin U\} \in I$.

Here, x_0 is called an I-limit of the sequence $\{x_n\}$ and denote as x_0 = I-lim x_n.

Theorem II.2. In a Cartesian product space X × Y , $\{(x_n, y_n)\}$ is a sequence of X × Y I-converges to $(x, y) \in X \times Y$ iff $\{x_n\}$ is a sequence in X I-converges to $x \in X$ and $\{y_n\}$ is a sequence in Y I-converges to $y \in Y$.

Proof. Suppose $\{(x_n, y_n)\}$is a sequence in $X \times Y$which is I-convergent to $(x, y) \in X \times Y$. Let Ube a open neighborhood of $x$and Vbe a open neighborhood of y.Then $U\times V$is the open neighborhood of (x, y).Since $\{(x_n, y_n)\}$ is sequence I-convergent to (x, y), $\{n \in \mathbb{N} : (x_n, y_n) \notin U \times V\} \in$I. Also,$\{n \in \mathbb{N} : x_n \notin U\} \subseteq \{n \in \mathbb{N} : (x_n, y_n) \notin U \times V\}$. Bythedefinitionofideal,$\{n\in\mathbb{N}: x_n\notin U\}\in$I.Similarly, we can obtain $\{n \in \mathbb{N} : y_n \notin V\} \in$I. Therefore, $\{x_n\}$ is I-convergent to x in X and $\{y_n\}$ is I-convergent to y in Y.

Conversely, suppose that $\{x_n\}$ is I-convergent to $x \in X$ and $\{y_n\}$ is I-convergent to y in Y. Let Gbe an open set containing an element (x, y). Since B = $\{U \times V : U \in \tau_X, V \in \tau_Y\}$ is a basisfor $X\times Y$,thereexists$U_x \in \tau_X$and$V_y \in \tau_Y$,suchthat $(x, y) \in U_x \times V_y \subseteq G$. Since $U_x$is open set in $X$and containing an element x, $\{n \in \mathbb{N}: x_n^{-1} \in U_x\} \in I$.Since $V_y$is open set in Y and containing an element y, $\{n \in \mathbb{N} : y_n \notin V_y\} \in$I. Then $\{n \in \mathbb{N} : (x_n, y_n) \notin U_x \times V_y\} = \{n \in \mathbb{N} : x_n \notin U_x\} \cup \{n \in \mathbb{N} : y_n \notin V_y\}$. Thisimplies,$\{n\in\mathbb{N}:(x_n,y_n)\notin U_x\times V_y\}\in$I.Therefore,$\{(x_n, y_n)\}$ is I-convergent to $(x, y) \in X \times Y$.

TheoremII.3.*Let(X,τ)beatopologicalspacewhichisfirstcountable.Ifevery*I-*convergentsequencehasexactlyone*I*-limit,thenXisHausdorff,here***I***mustbeanadmissibleideal.*

Proof. Let $x, y \in X$ and $x \neq y$. To prove X is Hausdorff, we have to find $U, V \in \tau$ such that, $x \in U, y \in Y$ and $U \cap V \neq \emptyset$. As X being first countable, we can find countable localbase for x and$y$.

Let $B_x = \{U_i \subset X: i \in \mathbb{N}\}andB_y = \{V_i \subset X: i \in \mathbb{N}\}$bethecountablelocalbaseforxandyrespectively.Define

Manuscript published on 30 December 2019.
* Correspondence Author (s)
B. Prakash*, Department of Mathematics, Kalasalingam Academy of Research and Education, Krishnankoil, India. Email: prakashphd101@gmail.com
B. Ananda priya, department, Department of Mathematics, Kalasalingam Academy of Research and Education, Krishnankoil, India. Email: xyz2@blueeyesintlligence.org

$B_i(x)=U_1\cap U_2\cap\cdots\cap U_i$ and $G_i(y)=V_1\cap V_2\cap\cdots\cap V_i, i\in\mathbb{N}$. Clearly, $B_i(x)$ and $G_i(y)$ areopensetssuchthat $x\in B_i(x)$ and $y\in G_i(y)$. Also, $B_j(x)\subseteq U_i$ and $G_j(y)\subseteq V_i$, forevery $i \leq j$.

Suppose $B_i(x)\cap G_i(y)=\varphi$, forsome i. Then $B_i(x)$ and $G_i(y)$ are disjoint open sets such that $x \in B_i(x)$ and $y \in G_i(y)$. Suppose $B_i(x)\cap G_i(y)\neq\varphi$, for any i. Then choose an element $x_i\in B_i(x) \cap G_i(y)$, $i \in\mathbb{N}$. This yields a sequence, say $\{x_n\}$. Now, we claim that, $\{x_n\}$ I-converges to x and y. Let U be any open set with $x \in U$. Since B_x is a countable base at x, there exists $U_n\in B_x$ such that $U_n\subseteq U$. Then $B_j(x) \subseteq U_n\subseteq U$, $\forall n \leq j$. This implies, $x_j\in B_j(x)\subseteq U_n\subseteq U, \forall n\leq j$. Therefore, $x_j\in U, \forall j\geq n$. Then, $\{n \in N : x_n\notin U \} \subseteq \{1, 2, \cdots , n - 1\}$. Since **I** is admissible, $\{n \in N : x_n\notin U \} \in I$. Therefore, $\{x_n\}$ I-converges to x. Let V be any open set with $y \in V$ .Since B_y is a countable base at y, there exists $V_n\in B_y$ such that $V_n\subseteq V$. Then $G_j(y)\subseteq V_n\subseteq V, \forall n\leq j$. Thisimplies, $x_j\in G_j(y)\subseteq V_n\subseteq V, \forall n\leq j$. Therefore, $x_j\in V, \forall j\geq n$. Then, $\{n\in N: x_n \notin V\}\subseteq\{1,2,\cdots,n-1\}$. **I** isadmissibleimplies $\{n\in N: x_n\notin V\}\in I$. Therefore, $\{x_n\}$ isI-convergent to y. Thus, $\{x_n\}$ I-converges to x and y. $\Rightarrow\Leftarrow$ as here every I-convergent sequence has unique I-limit. Therefore, $B_i(x) \cap G_i(y) = \varphi$ and so X isHausdorff.

III. I-SEQUENTIALLY COMPACT SPACES

In this section, we give some of the properties of I-sequentially compact spaces[1].

Theorem III.1. *The image of a I-sequentially compact spaces under the continuous map is also I-sequentially compact.*

Proof. Let $f : X \to Y$ be a continuous map and X be a I-sequentially compact space. We have to showthat $f(X)$ isanI-sequentiallycompactspace.Let $\{f(x_n)\}$ beanyarbitrarysequencein $f(X)$.Since X is I-sequentiallycompact, $\{x_n\}$ hasanI-clusterpoint,sayx.Let Vbeanopensetsuchthat $f(x)\in V$ and $\{f(x_n)\}$ beasequencein $f(X)$.Since f iscontinuous,thereexistsanopen set U containing x such that $U \subseteq f^{-1}(V)$.

Then $\{n \in N : x_n\in U \} \subseteq \{n \in N : x_n\in f^{-1}(V)\}$. Since U is an open set such that $x \in U$ and x is an I- cluster pointof $\{x_n\}$, $\{n \in N : x_n\in U \} \notin I$. We know that in an ideal I, if $I_1 \subseteq I_2$ and $I_1\notin I$, then $I_2\notin I$. Then $\{n \in N : x_n\in f^{-1}(V)\} \notin I$. This impliesthat, $\{ n \in N : f(x_n) \in V\} \notin I$. Therefore, f (x) is an I-cluster pointofthesequence $\{f(x_n)\}$.Thus, f(X) isI -sequentially compact.

Theorem III.2. Any closed subset of a I-sequentially compact space is I-sequentially compact.

Proof. Suppose Y is a closed subset of a I-sequentially compactspaceX.Let $\{y_n\}$ beanyarbitrarysequenceinY. Wehavetoprovethat $\{y_n\}$ hasanI-clusterpointinY.SinceX isI-sequentially compact and $y_n \in X$, $\{y_n\}$ has an I - cluster point, say y. Then, for every open set U with $y \in U$, $\{ n \in \mathbb{N} : y_n \in U \} \notin I$. Now we claim that $y \in Y$.

Then we have to consider the following two cases :

Case 1. Suppose y y_n.

Since $\{y_n\}$ is a sequence in Y, $y \in Y$.

Case 2. Suppose $y\notin\{y_n\}$.

Y is closed in X implies Y has all its accumulation points. Wehavetoprovethatyisanaccumulationpoint.Supposey isnotanaccumulationpointofY.Thenthere exists an open set V such that $y \in V$ and $Y \cap (V \setminus \{y\}) = \emptyset$. Then $\{y\}$ is open. This implies, $\{n \in \mathbb{N}: y_n \in V = \{y\}\} \notin I$. Therefore, $\emptyset \notin I$. $\Rightarrow\Leftarrow$ as here we assume I is an non-trivial ideal. Thus y is an accumulation point of Y and so that $y \in Y$. In both the cases, $y \in Y$. Therefore, y is an I-cluster point of the sequence $\{y_n\}$ in Y. Hence Y is I-sequentially compact.

Lemma III.3. Let (X, τ_1) and (X, τ_2) be any two completely regular space such that $\tau_1 \supset \tau_2$. If I is an admissible ideal, then the limit of the convergent sequence in (X, τ_2) and the I-cluster point of that sequence in (x, τ_1) are same.

Proof. Assume to the contrary that x is the limit point ofthe convergent sequence $\{x_n\}$ in (X, τ_2) and y is the I-cluster point of $\{x_n\}$ in (X, τ_1), x =y. Weknowthat,acompletelyregularspaceimpliesHausdorff space.SinceXisacompletelyregularspace,thereexistsopen sets U_x and U_y such that $x \in U_x, y \in U_y$ and $U_x \cap U_y = \varphi$ in X.Since $\{x_n\} \to x$, there exists $n_0\in\mathbb{N}$ such that $x_n \in U_x, \forall n \geq n_0$.

Then $\{n \in \mathbb{N}: x_n \in U_y\} \subset \{1,2,\ldots,n_0 - 1\} \in I$, since I is admissible. $\Rightarrow\Leftarrow$ as y is an I-cluster point (That is, $\{n \in \mathbb{N}: x_n \in U_y\} \notin I$.) Therefore, the limit of the convergent sequence in (X, τ_2) and the I-cluster point of that sequence in (X, τ_1) are same.

Theorem III.4. Let (X, τ_1) and (X, τ_2) be any two topological space such that

1) $\tau_1\supseteq\tau_2$
2) τ_1 is I-sequentiallycompact, I is an admissibleideal
3) τ_2 is completely regular and singleton sets are G_δ-sets

then $\tau_1 = \tau_2$.

Proof. Suppose V is a subset of X such that $V \notin\tau_2$. Then V is non-empty and so there exists a point $v \in V$ such that Visnotaneighborhoodofv.

As τ_2 is completely regular and singleton sets are G_δ-sets,there is a continuous function $g : (X, \tau_2) \to R$ such that $g^{-1}(0) = \{v\}$. Since V is not a neighborhood of v, for each $n \in \mathbb{N}$, the set $g^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$ is not wholly contained in V.Thus, foreach $n \in \mathbb{N}$, there is apoint $x_n \in X$ suchthat

$x_n \notin V$ and $|g(x_n)| < \frac{1}{n}$. Then $\{g(x_n)\} \to 0$.

As τ_1 is I-sequentially compact, $\{x_n\}$ has an I-cluster point y(say). By Theorem III.1, $\{g(x_n)\}$ has an I-cluster point g(y). Then by Lemma III.3, g(y) = 0. This implies that y = v. Therefore, $\{x_n\}$ has an I-cluster point v in (X, τ_1).

As $x_n \notin V$, for all n and $v \in V$, V is not a τ_1 −neighborhood of v. Thus $V \notin \tau_1$ and this implies that $\tau_2 \supset \tau_1$. Hence $\tau_1 = \tau_2$.

Theorem III.5. If $I_1 \subseteq I_2$ and if X is an

$\in\{$ $\}$

Retrieval Number: A12021291S419/2019©BEIESP
DOI:10.35940/ijeat.A1202.1291S419
Journal Website: www.ijeat.org

Published By:
Blue Eyes Intelligence Engineering
and Sciences Publication (BEIESP)

I_2-sequentiallycompactspace,thenXisanI_1-sequentiallycompactspace.

Proof.Let$\{x_n\}$beanysequenceinaI_2-sequentiallycompact space X. To prove X is I_1-sequentially compact, it isenough to prove that $\{x_n\}$ has an I_1-cluster point. Since X is an I_2-sequentially compact space and $\{x_n\}$ is a sequence inX,$\{x_n\}$has an I_2-cluster point, say x. Then, for every open setUsuchthatx$\in$U,$\{n\in N: x_n \in U\} \notin I_2$.Since$I_1 \subseteq I_2$,$\{n\in N: x_n \in U\} \notin I_1$.Therefore, $\{x_n\}$hasanI_1-clusterpoint x. Thus, X is I_1-sequentiallycompact.

REFERENCES

1. Amar Kumar Banerjee and Apurba Banerjee, A note on I-convergence and I*-convergent of sequences and nets in topological spaces, Matematicki Vesnik, **67** (3) (2015), 212-221.
2. V.Baláž,J.Červeňanský,P.KostyrkoandT.Šalát,I-convergenceand I-continuity of real functions, Faculty of Natural Sciences, Constantinethe Philosoper University, Nitra, **5** (2002), 43-50.
3. Benoy Kumar Lahiri and Pratulananda Das, Further results on I-limit superiorandI-limitinferior,Math.Commun.,**8**(2003),151-156.
4. Benoy Kumar Lahiri and Pratulananda Das, I and I*-convergence in topologicalspaces,MathematicaBohemica,**130**(2)(2005),153-160.
5. K. Demirci, I-limit superior and limit inferior, Math. Commun.,**6** (2001), 165-172.
6. H. Fast, Sur la convergence statistique, Colloq. Math., **2** (1951), 241-244.
7. P.Kostyrko,M.Mačaj,T.ŠalátandM.Sleziak,I-convergenceand extremal I-limit points, Math. Slovaca, **55** (4) (2005), 443-464.
8. P.Kostyrko,T.ŠalátandW.Wilczyński,I-convergence,RealAnalysis Exch., **26** (2) (2000), 669-685.
9. M.MačajandT.Šalát,Statisticalconvergenceofsubsequencesofa given sequence, Math. Bohem., **126** (2001), 191-208.
10. I. J. Schoenberg, The Integrability of certain function and related summability methods, Amer. Math. Monthly, **66** (1959),361-375.

AUTHORS PROFILE

Prakash currently working as a Assistant professor. He done his Ph.D on 2018. He published more that 5 papers in international and national journals. His research area is statistical sequence covering map and I-convergence.

Ananda Priya finished her M.Phil degree in Kalasalingam Academy of Research and Education. She currently doing research in the area of I-sequentially compact space.