

A Note on I-Sequentially Compact Spaces

Prakash B., Ananda Priya B.

Abstract—this paper investigates the properties of convergence and sequentially compact for the generalized concepts I -convergent and I -sequentially compact. We extend Cartesian product properties of convergence to I -convergence. Further, we develop some properties of I -sequential compactness of a topological space.

Keywords: Ideal, I -convergent sequence, I -cluster point, I -sequentially compact space.

I. INTRODUCTION

Convergent sequence in real numbers has been extended to statistical convergent separately by two different authors Fast [6] and Schoenberg [10]. Convergent sequence implies statistically convergent. But statistically convergent sequence need not be a convergent sequence. In general, statistically convergent sequences hold most of the properties of ordinary convergent sequences in metric spaces.

I -convergence of real sequences [7], [8] is a generalized concept of statistical convergence which is totally based on the structure of the ideal of subsets of the set of all natural numbers. In the recent literature, several works on I -convergence including remarkable contributions by Šalát et al. have occurred [2], [5], [7], [8], [3], [9]. I -convergence will become ordinary convergence if $I = \{I \subset N : n(I) < \infty\}$ and it will become statistical convergence if $I = \{I \subset N : d(I) = 0\}$ where $d(A) = \lim_{n \rightarrow \infty} \frac{| \{m \in A : m \leq n\} |}{n}$.

We recapitulate the following definitions.

If X is a non-empty set, then a family of sets $I \subset 2^X$ is called an **ideal** if

- 1) $I_1, I_2 \in I$ implies $I_1 \cup I_2 \in I$ and
- 2) $I_1 \in I, I_2 \subset I_1$ implies $I_2 \in I$.

The ideal I is called **nontrivial** if I is singleton set having \emptyset as an element and X is not an element of I . A non-trivial ideal I is called **admissible** if it contains all finite subsets of X . Several examples of non-trivial admissible ideal is given in [8]. Throughout this paper, we consider I as a non-trivial ideal of \mathbb{N} , the set of all natural numbers and (X, τ) denotes the topological space.

Also $\{x_n\}$ denote a sequence of elements of a topological space X . An element $y \in X$ is called an I -cluster point of

$\{x_n\}$ if $\{n \in \mathbb{N} : x_n \in U\} \notin I$, for every open set U containing y .

II. I-CONVERGENT SEQUENCE

Definition II.1. [4] Let (X, τ) be a topological space. A sequence $\{x_n\}$ is said to be I -convergent to an element $x_0 \in X$ if for every non-empty open neighborhood U containing $x_0, \{n \in \mathbb{N} : x_n \notin U\} \in I$.

Here, x_0 is called an I -limit of the sequence $\{x_n\}$ and denote as $x_0 = I\text{-lim } x_n$.

Theorem II.2. In a Cartesian product space $X \times Y$, $\{(x_n, y_n)\}$ is a sequence of $X \times Y$ I -converges to $(x, y) \in X \times Y$ iff $\{x_n\}$ is a sequence in X I -converges to $x \in X$ and $\{y_n\}$ is a sequence in Y I -converges to $y \in Y$.

Proof. Suppose $\{(x_n, y_n)\}$ is a sequence in $X \times Y$ which is I -convergent to $(x, y) \in X \times Y$. Let U be a open neighborhood of x and V be a open neighborhood of y . Then $U \times V$ is the open neighborhood of (x, y) . Since $\{(x_n, y_n)\}$ is sequence I -convergent to (x, y) , $\{n \in \mathbb{N} : (x_n, y_n) \notin U \times V\} \in I$. Also, $\{n \in \mathbb{N} : x_n \notin U\} \subseteq \{n \in \mathbb{N} : (x_n, y_n) \notin U \times V\}$. By the definition of ideal, $\{n \in \mathbb{N} : x_n \notin U\} \in I$. Similarly, we can obtain $\{n \in \mathbb{N} : y_n \notin V\} \in I$. Therefore, $\{x_n\}$ is I -convergent to x in X and $\{y_n\}$ is I -convergent to y in Y .

Conversely, suppose that $\{x_n\}$ is I -convergent to $x \in X$ and $\{y_n\}$ is I -convergent to y in Y . Let G be an open set containing an element (x, y) . Since $B = \{U \times V : U \in \tau_x, V \in \tau_y\}$ is a basis for $X \times Y$, there exists $U_x \in \tau_x$ and $V_y \in \tau_y$, such that $(x, y) \in U_x \times V_y \subseteq G$. Since U_x is open set in X and containing an element x , $\{n \in \mathbb{N} : x_n \notin U_x\} \in I$. Since V_y is open set in Y and containing an element y , $\{n \in \mathbb{N} : y_n \notin V_y\} \in I$. Then $\{n \in \mathbb{N} : (x_n, y_n) \notin U_x \times V_y\} = \{n \in \mathbb{N} : x_n \notin U_x\} \cup \{n \in \mathbb{N} : y_n \notin V_y\}$. This implies, $\{n \in \mathbb{N} : (x_n, y_n) \notin U_x \times V_y\} \in I$. Therefore, $\{(x_n, y_n)\}$ is I -convergent to $(x, y) \in X \times Y$.

Theorem II.3. Let (X, τ) be a topological space which is first countable. If every I -convergent sequence has exactly one I -limit, then X is Hausdorff, here I must be an admissible ideal.

Proof. Let $x, y \in X$ and $x \neq y$. To prove X is Hausdorff, we have to find $U, V \in \tau$ such that, $x \in U, y \in V$ and $U \cap V = \emptyset$. As X being first countable, we can find countable local base for x and y . Let $B_x = \{U_i \subset X : i \in \mathbb{N}\}$ and $B_y = \{V_i \subset X : i \in \mathbb{N}\}$ be the countable local base for x and y respectively. Define

$$B_i(x) = U_1 \cap U_2 \cap \dots \cap U_i \text{ and } B_i(y) = V_1 \cap V_2 \cap \dots \cap V_i, i \in \mathbb{N}$$

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* Correspondence Author

B. Prakash*, Department of Mathematics, Kalasalingam Academy of Research and Education, Krishnankoil, India. Email: prakashphd101@gmail.com

B. Ananda priya, department, Department of Mathematics, Kalasalingam Academy of Research and Education, Krishnankoil, India. Email: xyz2@blueeyesintelligence.org

Clearly, $B_i(x)$ and $G_i(y)$ are open sets such that $x \in B_i(x)$ and $y \in G_i(y)$. Also, $B_i(x) \subseteq U_i$ and $G_i(y) \subseteq V_i$, for every $i \leq j$.

Suppose $B_i(x) \cap G_i(y) = \emptyset$, for some i . Then $B_i(x)$ and $G_i(y)$ are disjoint open sets such that $x \in B_i(x)$ and $y \in G_i(y)$. Suppose $B_i(x) \cap G_i(y) \neq \emptyset$, for any i . Then choose an element $x_i \in B_i(x) \cap G_i(y)$, $i \in \mathbb{N}$. This yields a sequence, say $\{x_n\}$. Now, we claim that, $\{x_n\}$ I-converges to x and y . Let U be any open set with $x \in U$. Since B_x is a countable base at x , there exists $U_n \in B_x$ such that $U_n \subseteq U$. Then $B_j(x) \subseteq U_n \subseteq U$, $\forall n \leq j$. This implies, $x_j \in B_j(x) \subseteq U_n \subseteq U, \forall n \leq j$. Therefore, $x_j \in U, \forall j \geq n$. Then, $\{n \in \mathbb{N} : x_n \notin U\} \subseteq \{1, 2, \dots, n-1\}$. Since I is admissible, $\{n \in \mathbb{N} : x_n \notin U\} \in I$. Therefore, $\{x_n\}$ I-converges to x . Let V be any open set with $y \in V$. Since B_y is a countable base at y , there exists $V_n \in B_y$ such that $V_n \subseteq V$. Then $G_j(y) \subseteq V_n \subseteq V, \forall n \leq j$. This implies, $x_j \in G_j(y) \subseteq V_n \subseteq V, \forall n \leq j$. Therefore, $x_j \in V, \forall j \geq n$. Then, $\{n \in \mathbb{N} : x_n \notin V\} \subseteq \{1, 2, \dots, n-1\}$. I is admissible implies $\{n \in \mathbb{N} : x_n \notin V\} \in I$. Therefore, $\{x_n\}$ is I-convergent to y . Thus, $\{x_n\}$ I-converges to x and y . \Rightarrow as here every I-convergent sequence has unique I-limit. Therefore, $B_i(x) \cap G_i(y) = \emptyset$ and so X is Hausdorff.

III. I-SEQUENTIALLY COMPACT SPACES

In this section, we give some of the properties of I-sequentially compact spaces [1].

Theorem III.1. *The image of a I-sequentially compact spaces under the continuous map is also I-sequentially compact.*

Proof. Let $f : X \rightarrow Y$ be a continuous map and X be a I-sequentially compact space. We have to show that $f(X)$ is an I-sequentially compact space. Let $\{f(x_n)\}$ be any arbitrary sequence in $f(X)$. Since X is I-sequentially compact, $\{x_n\}$ has an I-cluster point, say x . Let V be any open set such that $f(x) \in V$ and $\{f(x_n)\}$ be a sequence in $f(X)$. Since f is continuous, there exists an open set U containing x such that $U \subseteq f^{-1}(V)$.

Then $\{n \in \mathbb{N} : x_n \in U\} \subseteq \{n \in \mathbb{N} : x_n \in f^{-1}(V)\}$. Since U is an open set such that $x \in U$ and x is an I-cluster point of $\{x_n\}$, $\{n \in \mathbb{N} : x_n \in U\} \notin I$. We know that in an ideal I , if $I_1 \subseteq I_2$ and $I_1 \notin I$, then $I_2 \notin I$. Then $\{n \in \mathbb{N} : x_n \in f^{-1}(V)\} \notin I$. This implies that, $\{n \in \mathbb{N} : f(x_n) \in V\} \notin I$. Therefore, $f(x)$ is an I-cluster point of the sequence $\{f(x_n)\}$. Thus, $f(X)$ is I-sequentially compact.

Theorem III.2. Any closed subset of a I-sequentially compact space is I-sequentially compact.

Proof. Suppose Y is a closed subset of a I-sequentially compact space X . Let $\{y_n\}$ be any arbitrary sequence in Y . We have to prove that $\{y_n\}$ has an I-cluster point in Y . Since X is I-sequentially compact and $y_n \in X$, $\{y_n\}$ has an I-cluster point, say y . Then, for every open set U with $y \in U$, $\{n \in \mathbb{N} : y_n \in U\} \notin I$. Now we claim that $y \in Y$. Then we have to consider the following two cases :

Case 1. Suppose $y = y_n$.

Since $\{y_n\}$ is a sequence in Y , $y \in Y$.

Case 2. Suppose $y \neq y_n$.

Y is closed in X implies Y has all its accumulation points. We have to prove that y is an accumulation point. Suppose

y is not an accumulation point of Y . Then there exists an open set V such that $y \in V$ and $Y \cap (V \setminus \{y\}) = \emptyset$.

Then $\{y\}$ is open. This implies, $\{n \in \mathbb{N} : y_n \in V = \{y\}\} \notin I$. Therefore, $\emptyset \notin I \Rightarrow$ as here we assume I is a non-trivial ideal. Thus y is an accumulation point of Y and so that $y \in Y$. In both the cases, $y \in Y$. Therefore, y is an I-cluster point of the sequence $\{y_n\}$ in Y . Hence Y is I-sequentially compact.

Lemma III.3. Let (X, τ_1) and (X, τ_2) be any two completely regular space such that $\tau_1 \supset \tau_2$. If I is an admissible ideal, then the limit of the convergent sequence in (X, τ_2) and the I-cluster point of that sequence in (X, τ_1) are same.

Proof. Assume to the contrary that x is the limit point of the convergent sequence $\{x_n\}$ in (X, τ_2) and y is the I-cluster point of $\{x_n\}$ in (X, τ_1) , $x \neq y$.

We know that, a completely regular space implies Hausdorff space. Since X is a completely regular space, there exists open sets U_x and U_y such that $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$ in X . Since $\{x_n\} \rightarrow x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U_x, \forall n \geq n_0$.

Then $\{n \in \mathbb{N} : x_n \in U_y\} \subseteq \{1, 2, \dots, n_0 - 1\} \in I$, since I is admissible. \Rightarrow as y is an I-cluster point (That is, $\{n \in \mathbb{N} : x_n \in U_y\} \notin I$.) Therefore, the limit of the convergent sequence in (X, τ_2) and the I-cluster point of that sequence in (X, τ_1) are same.

Theorem III.4. Let (X, τ_1) and (X, τ_2) be any two topological space such that

1) $\tau_1 \supset \tau_2$

2) τ_1 is I-sequentially compact, I is an admissible ideal

3) τ_2 is completely regular and singleton sets are G_δ -sets then $\tau_1 = \tau_2$.

Proof. Suppose V is a subset of X such that $V \notin \tau_2$. Then V is non-empty and so there exists a point $v \in V$ such that V is not a neighborhood of v .

As τ_2 is completely regular and singleton sets are G_δ -sets, there is a continuous function $g : (X, \tau_2) \rightarrow \mathbb{R}$ such that $g^{-1}(0) = \{v\}$. Since V is not a neighborhood of v , for each $n \in \mathbb{N}$, the set $g^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)$ is not wholly contained in V . Thus, for each $n \in \mathbb{N}$, there is a point $x_n \in X$ such that

$x_n \notin V$ and $|g(x_n)| < \frac{1}{n}$. Then $\{g(x_n)\} \rightarrow 0$.

As τ_1 is I-sequentially compact, $\{x_n\}$ has an I-cluster point y (say). By Theorem III.1, $\{g(x_n)\}$ has an I-cluster point $g(y)$. Then by Lemma III.3, $g(y) = 0$. This implies that $y = v$. Therefore, $\{x_n\}$ has an I-cluster point v in (X, τ_1) .

As $x_n \notin V$, for all n and $v \in V$, V is not a τ_1 -neighborhood of v . Thus $V \notin \tau_1$ and this implies that $\tau_2 \supset \tau_1$. Hence $\tau_1 = \tau_2$.

Theorem III.5. If $I_1 \subseteq I_2$ and if X is an I_2 -sequentially compact space, then X is an I_1 -sequentially compact space.

Proof. Let $\{x_n\}$ be any sequence in a I_2 -sequentially compact space X . To prove X is I_1 -sequentially compact, it is enough to prove that $\{x_n\}$ has an I_1 -cluster point. Since X is an I_2 -sequentially compact space and $\{x_n\}$ is a sequence in X , $\{x_n\}$ has an I_2 -cluster point, say x . Then, for every open set U such that $x \in U$, $\{n \in \mathbb{N} : x_n \in U\} \notin I_2$. Since $I_1 \subseteq I_2$, $\{n \in \mathbb{N} : x_n \in U\} \notin I_1$. Therefore, $\{x_n\}$ has an I_1 -cluster point x . Thus, X is I_1 -sequentially compact.

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AUTHORS PROFILE



Prakash currently working as a Assistant professor. He done his Ph.D on 2018. He published more that 5 papers in international and national journals. His research area is statistical sequence covering map and I-convergence.



Ananda Priya finished her M.Phil degree in Kalasalingam Academy of Research and Education. She currently doing research in the area of I-sequentially compact space.