Some Bounds on Greatest Common Divisor Degree Estrada Index of Graphs

K. Nagarajan, R. S. Ramkumar

Abstract - Let G be a simple graph of order n. In this paper we find some bounds on Greatest Common Divisor Degree Estrada Index of the graph G by using mathematical inequalities interms of G.C.D. degree Index I_{GCD} of graphs and exponential terms.

Index Terms – G.C.D. degree Energy, G.C.D. degree Estrada Index, G.C.D. degree spectral radius, G.C.D. degree Indexof graphs.

I. INTRODUCTION

In 1736, the concept of graph theory was developed by Euler. Graph theory is applied in all fields for solving combinatorial problems in various areas such as operations research, optimization, number theory and computer science. I.G.Gutman introduced a Energy of graphs in 1978. Spectral theory is a potential area of interdisciplinary research and Energy of graph is of recent interest. We introduced a greatest common degree energy of graphs and discussed about the greatest common divisor degree energy of some standard graphs[10]. Here we find some bounds for the greatest common divisor degree Estrada Index of graphs.

II. PRELIMINARIES

Here, we see some definitions and theoremsconcerned with energy of graphs and greatest common divisor energy of graphs for proving the main results in this paper.

Definition 2.1.[6] *Let G be a simple graph and let $V(G) = \{v_1, v_2, \dots, v_n\}$ be its vertex set. The adjacency matrix $A(G) = [a_{ij}]$ of the graph G is defined as a $n \times n$matrix where a_{ij} is 1 if the vertices v_i and v_j are adjacent or otherwise it is 0.*

Definition 2.2.[6] *Let G be a simple graph of order n and consider the adjacency matrix $A(G)$of the graph G. The eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A(G)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigen values of the graph G. Then the energy $E(G)$of G is defined as add up the absolute values of its eigen values that is $E(G) = \sum_{i=1}^{n} |\lambda_i|$.*

Definition 2.3. [10]*Let G be a simple graph and let $v_1, v_2, \dots, v_n$ be its vertices .Let degree of $v_i = d_i$ for every $i = 1,2,\dots,n$. Define*

$$a_{ij} = \begin{cases} g.c.d.(d_i, d_j) \text{ if } v_i, v_j \text{ areadjacent} \\ 0 \quad \text{otherwise} \end{cases}.$$

Then the square matrix $M(G) = [a_{ij}]$of order n is called a greatest common divisor degree matrix (G.C.D. degree matrix) of G. The characteristic polynomial of the G.C.D. degree matrix $M(G)$, denoted by $\phi(G;\lambda)$, is definedby $\phi(G;\lambda) = det(\lambda I - M(G)) = \lambda^n + c_1\lambda^{(n-1)} + \dots + c_n$ where I denote the $n \times n$ identity matrix. The roots$\lambda_1, \lambda_2, \dots, \lambda_n$,$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$,of the polynomial $\phi(G;\lambda)$are the greatest common divisor degree eigen values (G.C.D. degree eigen values) of G. The greatest common divisor degree energy (G.C.D. degree energy) of a graph G is $E_{GCD}(G) = \sum_{i=1}^{n} |\lambda_i|$. Among the G.C.D. degree eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$, the largest G.C.D. degree eigen value is called a G.C.D. degree spectral radius of G.

Example 2.4.*Consider a graph G. Here, $d(v_1) = 3$, $d(v_2) = 1, d(v_3) = 2, d(v_4) = 2$.*

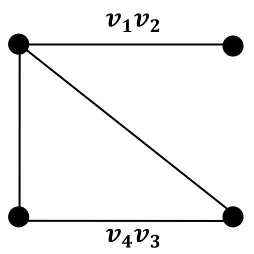

Fig 1

From fig.1, the G.C.D. degree matrix of the graph G is $M(G) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$. Then the G.C.D. degree eigen values are $\lambda_1 = -2, \lambda_2 \approx -1.343, \lambda_3 \approx 0.529$ and $\lambda_4 \approx 2.814$ and the greatest common divisor degree energy of the graph G is $E_{GCD}(G) \cong 6.686$.

Theorem 2.5. [9]*Assume that$G$ is a simple graph with G.C.D. degree index I_{GCD}. Then, $\lambda_1 \geq \frac{2I_{GCD}}{n}$ and also $\lambda_1 = \frac{2I_{GCD}}{n}$ if and only if G is k −G.C.D. degree regular.*

Definition 2.6. [11]Assume that G is a *(n,m)*-graph*which is simple and connected. Then the G.C.D. degree Estrada index of the graph $EE_{GCD}(G)$ is defined by $EE_{GCD}(G) = \sum_{i=1}^{n} e^{\lambda_i}$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are G.C.D. degree eigen values of G.*

$$\text{Let } N_t = \sum_{i=1}^{n} (\lambda_i)^t$$

$$\text{Then } EE_{GCD}(G) = \sum_{t \geq 0} \frac{N_t}{t!}$$

$$N_0 = \sum_{i=1}^{n} (\lambda_i)^0 = n$$

Revised Manuscript Received on December 15, 2019.

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$$N_1 = \sum_{i=1}^{n} (\lambda_i)^1 = trace\ of\ M(G) = 0$$

$$N_2 = \sum_{i=1}^{n} (\lambda_i)^2 = 2\left[\sum_{1\le i<j\le n} [g.c.d.(d_i, d_j)]^2\right]$$

III. BOUNDS ON GREATEST COMMON DIVISOR DEGREE ESTRADA INDEX OF GRAPHS

Here we find some bounds on G.C.D. degree Estrada index of graphs.

Theorem 3.1Assume that G is a *(n,m)*-graph*which is simple and connected. Then for any integer* $m_0 \ge 2$, $EE_{GCD}(G) \ge \sqrt{n^2 + \sum_{t=2}^{m_0} \frac{2^t N_t(G)}{t!}}$ *and* $EE_{GCD}(G)$ *becomes equal to its lower bound if and only if* $G \cong K_1$.

Proof:

Now $N_t(G)$ is equal to $[M(G)]^t$ where $M(G)$ is the G.C.D. degree matrix of G.

Thus $\sum_{t\ge m_0+1} \frac{2^t N_t(G)}{t!} \ge 0$

Then $\sum_{i=1}^{n} e^{2\lambda_i} = \sum_{i=1}^{n} \sum_{t\ge 0} \frac{(2\lambda_i)^t}{t!}$

$$= \sum_{t\ge 0} \frac{2^t N_t(G)}{t!}$$

$$\ge \sum_{t=0}^{m_0} \frac{2^t N_t(G)}{t!}$$

Also, $2\sum_{1\le i<j\le n} e^{\lambda_i} e^{\lambda_j} \ge n(n-1)\left(\prod_{i<j} e^{\lambda_i} e^{\lambda_j}\right)^{\frac{2}{n(n-1)}}$

$$= n(n-1)\left[\left(\prod_{i=1}^{n} e^{\lambda_i}\right)^{n-1}\right]^{\frac{2}{n(n-1)}}$$

$$= n(n-1)(e^{N_1})^{\frac{2}{n}}$$

$$= n(n-1)$$

Therefore, $[EE_{GCD}(G)]^2 = \sum_{i=1}^{n} e^{2\lambda_i} + 2\sum_{1\le i<j\le n} e^{\lambda_i} e^{\lambda_j}$

$$\ge n(n-1) + \sum_{t=0}^{m_0} \frac{2^t N_t(G)}{t!}$$

$EE_{GCD}(G) = \sqrt{n^2 + \sum_{t=2}^{m_0} \frac{2^t N_t(G)}{t!}}$ if and only if all the G.C.D. degree eigen values are 0 that is $G \cong K_1$.

Theorem 3.2 *Let* Assume that G is a graph*with* $n \ge 2$ *verticeswhich is simple and connected and G.C.D. degree index* I_{GCD}. *Then* $EE_{GCD}(G) \ge e^{\frac{2I_{GCD}}{n}} + (n-1)e^{\frac{-2I_{GCD}}{n(n-1)}}$ *and* $EE_{GCD}(G)$ *becomes equal to its lower bound if and only if* $G \cong K_1$.

Proof:

$$\text{Now, } EE_{GCD}(G) = e^{\lambda_i} + \sum_{i=2}^{n} e^{\lambda_i}$$

$$\ge e^{\lambda_1} + (n-1)\left(\prod_{i=2}^{n} e^{\lambda_i}\right)^{\frac{1}{n-1}}$$

$$= e^{\lambda_1} + (n-1)e^{\frac{\sum_{i=2}^{n}\lambda_i}{n-1}}$$

$$= e^{\lambda_1} + (n-1)e^{\frac{-\lambda_1}{n-1}}$$

$$\ge e^{\frac{2I_{GCD}}{n}} + (n-1)e^{\frac{-2I_{GCD}}{(n-1)}}$$

$EE_{GCD}(G) = e^{\frac{2I_{GCD}}{n}} + (n-1)e^{\frac{-2I_{GCD}}{n(n-1)}}$ if and only if all the G.C.D. degree eigen values are 0 that is $G \cong K_1$.

Theorem 3.3Assume that G is a *(n,m)*-graph*which is simple and connected. Then* $EE_{GCD}(G) \ge n - 2m$ *and* $EE_{GCD}(G) = n - 2m$ *if and only if* $G \cong K_1$.

Proof:

Now, $EE_{GCD}(G) = \sum_{i=1}^{n} \sum_{t\ge 0} \frac{(\lambda_i)^t}{t!}$

$$= n + 2\left[\sum_{1\le i<j\le n} [g.c.d.(d_i, d_j)]^2\right] + \sum_{i=1}^{n} \sum_{t\ge 2} \frac{(\lambda_i)^t}{t!}$$

As $e^{\lambda_i} \ge 1 + \lambda_i$holds for all$i$,$\sum_{t\ge 2} \frac{(\lambda_i)^t}{t!}$.

Let $\varepsilon \in [0,1]$.

Hence, $EE_{GCD}(G) \ge n + 2\left[\sum_{1\le i<j\le n} [g.c.d.(d_i, d_j)]^2\right] - n\varepsilon - 2\varepsilon\left[\sum_{1\le i<j\le n} [g.c.d.(d_i, d_j)]^2\right] + \varepsilon \sum_{i=1}^{n} \sum_{t\ge 0} \frac{(\lambda_i)^t}{t!}$

$$= (1-\varepsilon)n + 2(\varepsilon - 1)\left[\sum_{1\le i<j\le n} [g.c.d.(d_i, d_j)]^2\right] + EE_{GCD}(G)$$

Therefore$EE_{GCD}(G) \ge \frac{(1-\varepsilon)n + 2(\varepsilon-1)\left[\sum_{1\le i<j\le n}[g.c.d.(d_i,d_j)]^2\right]}{1-\varepsilon}$

$= n - 2\left[\sum_{1\le i<j\le n} [g.c.d.(d_i, d_j)]^2\right] \ge n - 2m.$

$EE_{GCD}(G) = n - 2m$if and only if all the G.C.D. degree eigen values are 0 that is $G \cong K_1$.

Theorem 3.4Assume that G is a *(n,m)*-graph*which is simple and connected.. Then* $EE_{GCD}(G) \le e[(n-1) + e^{E_{GCD}(G)}]$ *and* $EE_{GCD}(G) = e[(n-1) + e^{E_{GCD}(G)}]$ *if and only if* $G \cong K_1$.

Proof:

$$\text{Now} EE_{GCD}(G) = e\sum_{i=1}^{n} e^{\lambda_i - 1}$$

$$= e\sum_{i=1}^{n}\sum_{t\ge 0} \frac{(\lambda_i - 1)^t}{t!}$$

$$= e\left[n + \sum_{i=1}^{n}\sum_{t\ge 0} \frac{(\lambda_i - 1)^t}{t!}\right]$$

$$\le e\left[n + \sum_{i=1}^{n}\sum_{t\ge 0} \frac{(|\lambda_i - 1|)^t}{t!}\right]$$

$$\le e\left[n + \sum_{t\ge 0} \frac{(\sum_{i=1}^{n}|\lambda_i - 1|)^t}{t!}\right]$$

$$\le e\left[n - 1 + \sum_{t\ge 0} \frac{(\sum_{i=1}^{n}|\lambda_i|)^t}{t!}\right]$$

$$= e\left[n - 1 + e^{E_{GCD}(G)}\right]$$

$EE_{GCD}(G) = e[(n-1) + e^{E_{GCD}(G)}]$if and only if all the G.C.D. degree eigen values are 0 that is $G \cong K_1$.

Theorem 3.5Assume that G is a *(n,m)*-graph*which is simple and connected. Then*

$$EE_{GCD}(G) \geq n + m + \frac{1}{2e}[e^2 - 2e - 1]N_3 + \frac{1}{2e}[e^2 - 3e - 1]N_4$$

Proof:

Now, $EE_{GCD}(G) = N_0 + N_1 + N_2 + \sum_{t\geq 3} \frac{N_t}{t!}$

$$\geq n + 0 + \frac{2}{2!}\left[\sum_{1\leq i<j\leq n} [g.c.d.(d_i, d_j)]^2\right] + \sum_{t\geq 3} \frac{N_t}{t!}$$

$$\geq n + m + \sum_{t\geq 2} \frac{N_{2t-1}}{(2t-1)!} + \sum_{t\geq 2} \frac{N_{2t}}{(2t)!}$$

$$\geq n + m + N_3 \sum_{t\geq 2} \frac{1}{(2t-1)!} + N_4 \sum_{t\geq 2} \frac{1}{(2t)!}$$

$$= n + m + N_3\left(\frac{e - e^{-1}}{2} - 1\right) + N_4\left(\frac{e + e^{-1}}{2} - 1 - \frac{1}{2!}\right)$$

$$= n + m + \frac{1}{2e}[e^2 - 2e - 1]N_3 + \frac{1}{2e}[e^2 - 3e - 1]N_4$$

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