

# Pell Labeling of Joins of Square of Path Graph

S. Sriram, R. Govindarajan, K. Thirusangu

**Abstract:** A graph is composed of  $p$  vertices and  $q$  edges. A Pell labeling graph is the one with  $u \in V(G)$  being distinct. Label  $f(u)$  from  $0, 1, 2, \dots, p-1$  in a such a way that each edge is labelled with  $f^* : E(G) \rightarrow N$  such that  $f^*(uv) = f(u) + 2f(v)$  are distinct. In this paper we study Square of Path graph  $P_n^2$  and attach an edge to form a join to the square of path graph  $P_n^2$  and prove the join of square of path graph  $P_n^2$  is Pell labelling graph and further study on some interesting results connecting them.

**Keywords :** Square of Path graph  $P_n^2$ , Pell labelling , Pell labelling graph, Joins of Square of path graph  $P_n^2$

## I. INTRODUCTION

A finite graph has finite vertices and finite edges. Gallian [1] has provided an interesting survey on graph labeling. Rosa [2] has initiated the study on labeling. Pell labelling of graph was introduced by J. Shiama [3] and have proved that paths, cycles, stars, double stars, coconut tree, bi star are Pell labelling graphs. Motivated towards the Pell Labeling graph and study of joins of graphs [4][5][6][7] we in this paper have identified the Square of path graph  $P_n^2$  and proceeded further to study on some important results. For graph preliminaries we consider Gross.J and .Yellen.J, Handbook of graph theory [8].

## II. PRELIMINARIES

**Definition 2.1:** A Pell labelling graph is a bijection function  $f : V(G) \rightarrow \{0, 1, 2, \dots, p-1\}$  such that for each edge there is an induced distinct edge labelling  $f^* : E(G) \rightarrow N$  such that  $f^*(uv) = f(u) + 2f(v)$ .

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**Definition 2.2:** A path graph  $P_n$  consists of vertices  $u_1, u_2, \dots, u_n$  the square of path  $P_n^2$  graph consists  $n$  vertices  $u_1, u_2, \dots, u_n$  and  $2n-3$  edges

**Definition 2.3 :** Attaching a square of path  $P_n^2$  graph with another square of path  $P_n^2$  graph by an edge is called 1-join square of path  $P_n^2$  graph.

**Note:** For 1-join square of path  $P_n^2$  graph the number of vertices is  $2n$  and the number of edges is  $4n-5$

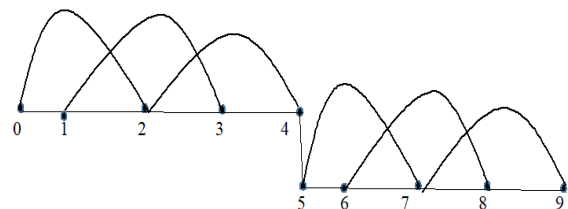


Fig.1- 1-join of square of path  $P_5^2$

The above is a 1-join of square of path  $P_5^2$  graph with 5 vertices and 7 edges attached by an edge to one square path graph  $P_5^2$  with another square path graph  $P_5^2$ . Similarly we can construct M-Join of square of path  $P_n^2$  graph.

**Note:** We here exclusively study the case of M-Join of square of Path  $P_n^2$  graph attached by edges  $e_1, e_2, \dots, e_n$  to the same order of square of path  $P_n^2$  graph.

## III. MAIN RESULTS

**Theorem .3.1:** 1- Join square of path  $P_n^2$  is a Pell labelling graph.

**Proof:** Let  $G=1$ -Join of square of path  $P_n^2$  graph

Let us prove that  $G$  is a Pell labeling graph

Let us prove the theorem by labelling the vertices of the graph  $G$ .

We have the number of vertices in square of path  $P_n^2$  graph is  $n$  and the number of edges is  $3n-2$ . Now adding one edge between the square of path  $P_n^2$  graph with another square of path  $P_n^2$  graph we have the number of vertices in 1-Join

Of square of path  $P_n^2$  graph is  $2n$  and the number of edges in 1-Join of square of path graph is  $4n-5$ .

Now let us label the vertices of 1-join of square of path  $P_n^2$  graph as follows We know that the first square of path  $P_n^2$  has  $n$  vertices and second square of path  $P_n^2$  has  $n$  vertices and totally  $2n$  vertices is to be labelled.

Let us denote the Vertex Set of first square of path  $P_n^2$  graph as  $V = \{u_1, u_2, u_3, u_4, \dots, u_n\}$  and the vertex set of second square of path  $P_n^2$  graph as  $V^1 = \{u_1^1, u_2^1, u_3^1, u_4^1, \dots, u_n^1\}$ . The edge set of first square of path  $P_n^2$  graph is  $E = \{e_1, e_2, e_3, e_4, \dots, e_{n-1}\} \cup$

$$\{e_{ij}, 1 \leq i \leq n-2, n-2 \leq j \leq n\}$$

and the edge of the second square of path  $P_n^2$  Graph is

$$E^1 = \{e_1^1, e_2^1, e_3^1, e_4^1, \dots, e_{n-1}^1\} \cup$$

$$\{e_{ij}^1, 1 \leq i \leq n-2, n-2 \leq j \leq n\}.$$

Now by adding one edge between the first and second square of path  $P_n^2$  graph we have the vertex set of 1-Join square of path  $P_n^2$  graph as

$$V(G) = \{u_1, u_2, u_3, u_4, \dots, u_n\} \cup \{u_1^1, u_2^1, u_3^1, u_4^1, \dots, u_n^1\}$$

and the edge set as  $E(G) = \{e_1, e_2, e_3, e_4, \dots, e_{n-1}\} \cup$

$$\{e_1^1, e_2^1, e_3^1, e_4^1, \dots, e_{n-1}^1\} \cup$$

$$\{e_{ij}, e_{ij}^1, 1 \leq i \leq n-2, n-2 \leq j \leq n\} \cup \{e\}.$$

The edges that are connecting between the first and second square of path  $P_n^2$  graph is known as

$$e_i = u_i u_{i+1} \text{ for } 1 \leq i \leq n-1$$

$$e_i^1 = u_i^1 u_{i+1}^1 \text{ for } 1 \leq i \leq n-1$$

$$e_{ij} = u_i u_j \text{ for } 1 \leq i \leq n-2 \text{ and for } n-2 \leq j \leq n$$

$$e_{ij}^1 = u_i^1 u_j^1 \text{ for } 1 \leq i \leq n-2 \text{ and for } n-2 \leq j \leq n$$

$$e = e_{n-1} e_1^1$$

Now let us label the vertices of first square of path  $P_n^2$  graph in correspondence to the vertices of second square of path  $P_n^2$  graph as follows

$$f(u_i) = i \text{ for } 0 \leq i \leq n-1$$

$$f(u_j^1) = (n-1) + j \text{ for } 1 \leq j \leq n$$

Computing the induced edge labelling we have

$$f^*(e_i) = f^*(u_i u_{i+1}) = f(u_i) + 2f(u_{i+1})$$

$$f^*(e_i^1) = f^*(u_i^1 u_{i+1}^1) = f(u_i^1) + 2f(u_{i+1}^1)$$

$$f^*(e_{ij}) = f^*(u_i u_j) = f(u_i) + 2f(u_j)$$

$$f^*(e_{ij}^1) = f^*(u_i^1 u_j^1) = f(u_i^1) + 2f(u_j^1)$$

$$f^*(e) = f^*(u_{n-1} u_1^1) = f(u_{n-1}) + 2f(u_1^1)$$

The induced edge labelling is distinct Hence the proof.

We now give a general algorithm for assigning labels for the vertices of M-Join of square of path  $P_n^2$  graph

**Algorithm.3.2**

Begin

for  $j=0$  to  $n-1$

$$f(u_{j+1}) = j$$

for  $i = 1$  to  $M$

for  $j= 0$  to  $n-1$

$$f(u_{j+1}^i) = i(n) + j$$

End

**Note.1:** Following the algorithm to label the vertices of M-Join of square of path  $P_n^2$  graph we can obtain the induced edge labelling of the graph and hence can prove in general for any M-Join of square of path  $P_n^2$  graph is Pell labelling graph.

**Note.2:** We know from basic algebra that if three numbers

$a, b, c$  are in arithmetic progression then  $b = \frac{a+c}{2}$  or

otherwise  $b-a = c-b$ . Let us use this technique in the subsequent theorem to prove that the induced edge labelling of M-Join of square of path  $P_n^2$  graph forms an arithmetic progression.

**Theorem.3.3:** M-join of square of path  $P_n^2$  Pell graph the induced edge labelling has the following property

(i)  $f^*(u_i u_{i+1}), f^*(u_{i+1} u_{i+2}) \dots$  are in arithmetic progression with common difference 3 by fixing  $i = 1$

$f^*(u_i^M u_{i+1}^M), f^*(u_{i+1}^M u_{i+2}^M) \dots$  are in arithmetic progression with common difference 3 where  $M=1, 2, \dots$  and  $M$  is the number of joins of square of path  $P_n^2$  graph and by fixing  $i = 1$

(ii)  $f^*(u_i u_{i+2}), f^*(u_{i+1} u_{i+3}) \dots$  are in arithmetic progression with common difference 3 by fixing  $i = 1$ .

$f^*(u_i^M u_{i+2}^M), f^*(u_{i+1}^M u_{i+3}^M) \dots$  are in arithmetic progression with common difference 3 by fixing  $i = 1$ .

**Proof:** Consider M-Join of square of path  $P_n^2$  Pell graph.

Now labelling the graph as given in the algorithm we can obtain the induced edge labelling case by case for different joins of square of path  $P_n^2$  graph.

Now let us prove the property (i) as follows

To prove  $f^*(u_i u_{i+1}), f^*(u_{i+1} u_{i+2}) \dots$  are in arithmetic progression. let us consider

$$a = f^*(u_i u_{i+1}) = f(u_i) + 2f(u_{i+1})$$

$$b = f^*(u_{i+1}u_{i+2}) = f(u_{i+1}) + 2f(u_{i+2})$$

$$c = f^*(u_{i+2}u_{i+3}) = f(u_{i+2}) + 2f(u_{i+3})$$

Now let us use the property that  $a, b, c$  are in Arithmetic progression namely  $b - a = c - b$

On computing

$$b - a = f^*(u_{i+1}u_{i+2}) - f^*(u_i u_{i+1}) = (f(u_{i+1}) + 2f(u_{i+2}))$$

$$- (f(u_i) + 2f(u_{i+1}))$$

Simplifying the terms

$$b - a = 2f(u_{i+2}) - f(u_i) - f(u_{i+1})$$

Similarly computing

$$c - b = f^*(u_{i+2}u_{i+3}) - f^*(u_{i+1}u_{i+2}) = f(u_{i+2}) + 2f(u_{i+3})$$

$$- f(u_{i+1}) - 2f(u_{i+2})$$

Simplifying the terms

$$c - b = 2f(u_{i+3}) - f(u_{i+1}) - f(u_{i+2})$$

Now equating  $b - a = c - b$  and simplifying we have

$$2f(u_{i+2}) - f(u_i) = 2f(u_{i+3}) - f(u_{i+2})$$

On further simplifying we find

$$3f(u_{i+2}) = f(u_i) + 2f(u_{i+3})$$

Which is the condition required for the induced edge labelling  $f^*(u_i u_{i+1}), f^*(u_{i+1} u_{i+2}) \dots$  to be in arithmetic progression. To verify it let us assume  $i=1$  so as to identify the labels of the vertices in the condition on substituting in the condition we find the terms are

$$f(u_1) = 0; f(u_3) = 2; f(u_4) = 3$$

Hence on substituting in the condition the labels we find that it is 6 on both sides. Similarly we can assign values for  $i$  as 2, 3, ... and verify that the induced edge labelling are in arithmetic progression.

In similar way we can prove  $f^*(u_i u_{i+2}), f^*(u_{i+1} u_{i+3}) \dots$  are in arithmetic progression. Similarly we can prove result (ii) namely  $f^*(u_i u_{i+2}), f^*(u_{i+1} u_{i+3}) \dots$  and

$$f^*(u_i^M u_{i+2}^M), f^*(u_{i+1}^M u_{i+3}^M) \dots$$

are in arithmetic progression. We find from the labelling techniques that the common difference is 3 for both result (i) and result (ii). Hence the proof.

### Example.1

The following is the 1-Join of square of path  $P_5^2$  graph. we label as in theorem.3.2

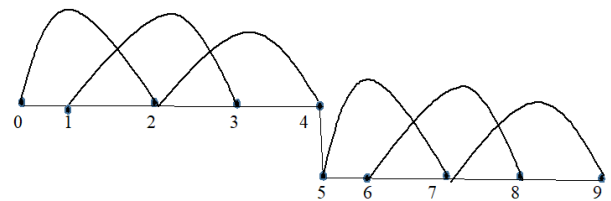


Fig.2- 1-join of square of path  $P_5^2$

For Result .(i) is true i.e  $f^*(u_i u_{i+1}), f^*(u_{i+1} u_{i+2}) \dots$  are in arithmetic progression with common difference 3.

For we know the labelling of 1-join of square of path  $P_5^2$  graph is as follows

$$f(u_1) = 0, f(u_2) = 1, f(u_3) = 2, f(u_4) = 3, f(u_5) = 4, f(u_1^1) = 5, f(u_2^1) = 6, f(u_3^1) = 7, f(u_4^1) = 8, f(u_5^1) = 9$$

The induced edge labelling of the graph is given as

$$f^*(u_1 u_2) = 2; f^*(u_2 u_3) = 5; f^*(u_3 u_4) = 8; f^*(u_4 u_5) = 11; f^*(u_1^1 u_2^1) = 17; f^*(u_2^1 u_3^1) = 20; f^*(u_3^1 u_4^1) = 23; f^*(u_4^1 u_5^1) = 26$$

The above induced edge labelling proves the result.(i) in theorem.3.2

Also let us compute the induced edges labelling in order to prove the result .(ii) of theorem.3.2 namely

$$f^*(u_i u_{i+2}), f^*(u_{i+1} u_{i+3}) \dots \text{ and for the joins } f^*(u_i^M u_{i+2}^M), f^*(u_{i+1}^M u_{i+3}^M) \dots \text{ as } f^*(u_1 u_3) = 4; f^*(u_2 u_4) = 7; f^*(u_3 u_5) = 10; f^*(u_1^1 u_3^1) = 19; f^*(u_2^1 u_4^1) = 22; f^*(u_3^1 u_5^1) = 25$$

The above induced edge labelling proves the result .(ii) in theorem.3.2.

**Note:** Similarly for M-Join of square of path  $P_n^2$  Pell the result is true.

**Theorem.3.4:** M-Join of square of path  $P_n^2$  Pell graph the following property holds good

$$f^*(u_n u_1^M) = f^*(u_{n-2} u_n) + 4, \text{ for } M=1 \text{ and } , \text{ for } M \geq 2 \text{ where } M \text{ is the number of joins of square of path } P_n^2 \text{ graph and } n \text{ is the number of vertices of square of path } P_n^2 \text{ graph.}$$

**Proof :** Let us consider M-Join of square of path  $P_n^2$  Pell graph following the labelling techniques provided in Theorem.3.1 and stated in the Algorithm. For proving the result let us choose  $n=3$  and  $M=1$ .

$$\text{Now to Prove } f^*(u_n u_1^M) = f^*(u_{n-2} u_n) + 4$$

Let us substitute  $n=3$  and  $M=1$  and obtain as follows

$$f^*(u_3 u_1^1) = f^*(u_1 u_3) + 4$$

$$\text{Now } f^*(u_3 u_1^1) = f(u_3) + 2f(u_1^1) = 2 + 2(3) = 8$$

Also

$$f^*(u_1 u_3) + 4 = f(u_1) + 2f(u_3) + 4 = 0 + 2(2) + 4 = 8$$

Hence the result is true for  $n=3$  and  $M=1$ . Similarly we can prove for any  $n$  and for  $M=1$  in a similar fashion by following the schema of labelling of vertices .

The result  $f^*(u_{n-2}^M u_n^M) = f^*(u_{n-2}^{M-1} u_n^{M-1}) + 4$  can also be proved for  $M \geq 2$ . Hence the proof.

**Theorem.3.5:** M-Join of square of path  $P_n^2$  Pell graph the following property holds good

$$(i) f^*(u_i u_{i+2}) = (f(u_i) + f(u_{i+1}) + f(u_{i+2})) + 1, \quad ,$$

$$f^*(u_i^M u_{i+2}^M) = (f(u_i^M) + f(u_{i+1}^M) + f(u_{i+2}^M)) + 1, \quad ,$$

where  $M$  is the number of joins and  $i = 1, 2, \dots, n-2$  of the square of path  $P_n^2$  graph.

$$(ii) f^*(u_i u_{i+2}) = f^*(u_i u_{i+1}) + 2, \quad ,$$

$$f^*(u_i^M u_{i+2}^M) = f^*(u_i^M u_{i+1}^M) + 2 \text{ and}$$

$$(iii) f^*(u_{i+1} u_{i+2}) = f^*(u_i u_{i+2}) + 1, \quad ,$$

$$f^*(u_{i+1}^M u_{i+2}^M) = f^*(u_i^M u_{i+2}^M) + 1,$$

**Proof:** Consider M-Join of square of path  $P_n^2$  Pell graph with the labelling techniques given in theorem.3.1 and algorithm. Now to prove the property as stated in the theorem. Let us choose  $i=1$

Now let us substitute  $i=1$  in

$$f^*(u_i u_{i+2}) = (f(u_i) + f(u_{i+1}) + f(u_{i+2})) + 1$$

Which is computed as follows

$$f^*(u_1 u_3) = (f(u_1) + f(u_2) + f(u_3)) + 1$$

$$\text{We know } f^*(u_1 u_3) = f(u_1) + 2f(u_3)$$

On substituting the labels for the corresponding vertices we have

$$f^*(u_1 u_3) = f(u_1) + 2f(u_3) = 0 + 2(2) = 4$$

Now computing the R.H.S of the result as follows

$$f(u_1) + f(u_2) + f(u_3) + 1 = 0 + 1 + 2 + 1 = 4$$

Hence we find that the result

$$f^*(u_i u_{i+2}) = (f(u_i) + f(u_{i+1}) + f(u_{i+2})) + 1 \text{ is true}$$

for  $i=1$ .

Similarly it can be proved for any value of  $i$ . Hence the result (i) of the theorem.

Now let us prove result (ii) by choosing  $i=1$

To prove  $f^*(u_i u_{i+2}) = f^*(u_i u_{i+1}) + 2$ . Let us substitute  $i=1$  and hence find the following

$$f^*(u_i u_{i+2}) = f^*(u_i u_{i+1}) + 2$$

On substituting the corresponding labels for the vertices we have

$$f^*(u_1 u_3) = f(u_1) + 2f(u_3) = 0 + 2(2) = 4$$

The R.H. S of the result is computed as follows

$$f^*(u_1 u_2) + 2 = f(u_1) + 2f(u_2) + 2$$

On substituting the corresponding labels for the vertices we have

$$f^*(u_1 u_2) + 2 = f(u_1) + 2f(u_2) + 2 = 0 + 2(1) + 2 = 4$$

Hence we find the result  $f^*(u_i u_{i+2}) = f^*(u_i u_{i+1}) + 2$  is true for  $i=1$ . Hence the result (ii) of the theorem.

Now to claim the result (iii) of the theorem namely

$$f^*(u_{i+1} u_{i+2}) = f^*(u_i u_{i+2}) + 1$$

Let us choose  $i=1$  and compute L.H.S and R.H.S as follows  
L.H.S is given by

$$f^*(u_2 u_3) = f(u_2) + 2f(u_3) = 1 + 2(2) = 5$$

R.H.S is given by

$$f^*(u_1 u_3) + 1 = f(u_1) + 2f(u_3) + 1 = 0 + 2(2) + 1 = 5$$

Hence it is evident that result (iii) is also true for  $i=1$ . In general the result (i) (ii) and (iii) is all true for all values of  $i$  and for all values of  $M$  by proceeding in a similar way. Hence the theorem.

#### IV. CONCLUSION

We are studying on different graphs in a similar approach to generalize the results

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#### REFERENCES

- Gallian . J.A , A Dynamic Survey of Graph Labeling,2018;Twenty first edition
- Rosa.A, On certain valuation of the vertices of a graph in Theory of Graphs, (Intl.Symp.Rome.1966), Gordon and Breach, Dunod Paris,1967,349-355.
- Shiama .J , Pell Labeling of some graphs, Asian Journal of Current Engineering and Maths, Vol 2, No 4 , 2013
- Sriram .S, Govindarajan R, Super Harmonic Mean Labeling of Joins of Square of Path Graph, International Journal of Research in Advent Technology, Vol.7, No.3, March , 2019
- Sharon Philomena V, Thirusangu K, 3-Total Product Cordial and Pell Labeling on Tree  $\langle K_{1,n} : 2 \rangle$ , International Journal of Pure and Applied Mathematics, Vol. 101 No.5, pp.747-752, 2015
- Muthuramakrishnan D and Sutha S, Pell graceful labeling of graphs, Malaya Journal of Matematik, Vol.7, No.3, 508-512,2019

7. Avudainayaki R, Selvam B, Harmonious and Pell Labeling of Some Extended Duplicate Graph, International Journal of Scientific Research in Mathematics and Statistical Sciences, Vol.5, Issue.6, pp.152-156,2018
8. Gross.J and .Yellen.J, Handbook of graph theory, CRC, Press: 2004

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