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Zumkeller Labeling of Complete Graphs

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Abstract**: *Let G (V, E) be a graph with vertex set V and edge set E. The process of assigning natural numbers to the vertices of G such that the product of the numbers of adjacent vertices of G is a Zumkeller number on the edges of G is known as Zumkeller labeling of G. This can be achieved by defining an appropriate vertex function of G. In this article, we show the existence of this labeling to complete graph and fan graph.***

Keywords**: *Zumkeller Number, Labeling, Injective Function and Complete Graph, Fan Graph.***

I. INTRODUCTION

In the theory of graph labeling, the labels are computed to vertices or edges of the graph through mathematical functions. In the year 1960, Alex Rosa[13] had introduced first the graph labeling concept. For the recent research trends in graph labeling and history of graph labeling one can refer [7]. The notations and terminology of graph theory used in this paper are referred to Harary [9]. The applications of labelled graphs have found in recent research.

The applications of graceful labeling of complete graph to coding theory are presented in [8]. Krishnappa H.K et.al in [11, 12] showed the practical applications of the complete graph K_n in the field of cryptography. Zongheng et al. [15] discussed the connectivity and coverage problems in sensor network. Various algorithms were presented for the selection of minimum number of nodes for a communication graph which results in different graph topologies. Jonathann et al. [10] explored the applications of graph labeling in network coding related to mobile communications.
The Zumkeller graphs were introduced by Balamurugan et al in [3] in the year 2014. Also in the literature [1-6],the Zumkeller graphs, Strongly multiplicative Zumkeller graphs and k-Zumkeller graphs have been introduced and investigated by Balamurugan et al. In this paper, we investigate the existence of Zumkeller complete graphs and Zumkeller fan graphs.

II. PRELIMINARIES

We refer the paper [14] for the concept and properties of Zumkeller numbers.

Definition 2. 1.

A Zumkeller number n is a real positive integer in which the positive factors of n can be partitioned into two disjoint sets Such that the sum of the integers in the first set is equal to the sum of the integers in the second set.
This partition of n is known as Zumkeller partition.

Example 2.1

40 is a Zumkeller number because the positive factors of 40 can be partitioned into A={5,40} and B={1,2,4,8,10,20} where the sum of each set is 45.

6 is the first Zumkeller number and other few Zumkeller numbers are 12, 20, 24, 28, 30, ….whose partitions are as follows:

6	[6]	[1, 2, 3]
12	[2, 12]	[1, 3, 4, 6]
20	[1, 20]	[2, 4, 5, 10]
24	[6, 24]	[1, 2, 3, 4, 8, 12]
28	[28]	[1, 2, 4, 7, 14]
30	[6, 30]	[1, 2, 3, 5, 10, 15]

Remark

The numbers which are perfect square or twice the perfect square are not Zumkeller numbers.

III. MAIN RESULTS

In this section, we prove the complete graph K_n, one point union of two complete graphs $K_n^{(2)}$ and fan graph $F_n = P_n + K_1$ admit the Zumkeller labeling.

Definition3.1

A simple graph $G = (V, E)$, where V is vertex set and E is edge set of G , is said to admit a Zumkeller labeling if there exists an injective function $f : V \to N$ such that the induced edge function $f^* : E \to N$ defined as $f^*(xy) = f(x) f(y)$ is a Zumkeller number for $xy \in E$; $x, y \in V$. The labelled graph G is called as a Zumkeller graph.

Example 3.1

The graph shown in the Fig.2.1 is a Zumkeller graph.

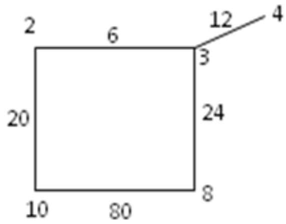

Fig.2.1: A Zumkeller Graph

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Definition3.2

A simple graph in which every pair of vertices adjacent is called a complete graph.

Theorem 3.1:

The complete graph K_n is a Zumkeller graph when $n \equiv 2(\bmod 4)$ and $n \leq 10$.

Proof:

Let $V= \{v_i \mid 1\leq i \leq n\}$ be the vertex set and $E = \{e_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n \ and \ i \leq j\}$ be the edge set of the complete graph K_n.

Define an injective function $f: V \to N$ such that

$$f(v_1) = \frac{n}{2}\left(\frac{n}{2}+1\right)$$

$$f(v_i) = \left(\frac{n}{2}+i-1\right)\left(\frac{n}{2}+i\right) \text{ for } 2 \leq i \leq n$$

Define an induced function $f^*: E \to N$ such that $f^*(e_{ij}) = f^*(v_i v_j) = f(v_i) f(v_j)$.

Let $f(v_i) = a \times b$ and $f(v_j) = c \times d$ where a, b and c, d are a pair of consecutive integers.

Claim: To prove $f(v_i) f(v_j) = abcd$ is a Zumkeller number.

Now,

Since $n \leq 10$,
$$f(v_n) = \left(\frac{n}{2}+n-1\right)\left(\frac{n}{2}+n\right)$$

We have $f(v_n) \leq$ 14x15

Here the integers in the product of two consecutive integers are less than or equal to 15. All odd numbers other than 9 and 15 are primes in this product and in each pair of consecutive numbers there is an odd number.

Therefore, the product $abcd$ of those numbers have the prime factorisation of the form $2^k p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ for which at least one k_i is odd.

Now we prove that the products of any two numbers in the set $\{8\times 9, 9\times 10, 14\times 15\}$ are also the same form of prime factorisation.

$$(i) \quad 8\times 9\times 9\times 10 = 2^4 3^4 5,$$

$$(ii) \quad 8\times 9\times 14\times 15 = 2^4 3^3 5$$

$$(iii) \quad 9\times 10\times 14\times 15 = 2^2 3^3 5 7$$

Therefore, all the edges in the complete graph K_n have the prime factorisation of the form $2^k p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ in which at least one k_i is odd.

By the property of Zumkeller numbers, *abcd* is a Zumkeller number.

Hence the complete graph K_n admits a Zumkeller labeling when $n \equiv 2(\bmod 4)$ and $n \leq 10$.

An illustration of this theorem is shown in the example 3.1.

Example 3.2

The graph shown in the Fig.3.1 is a Zumkeller labeling of K_6.

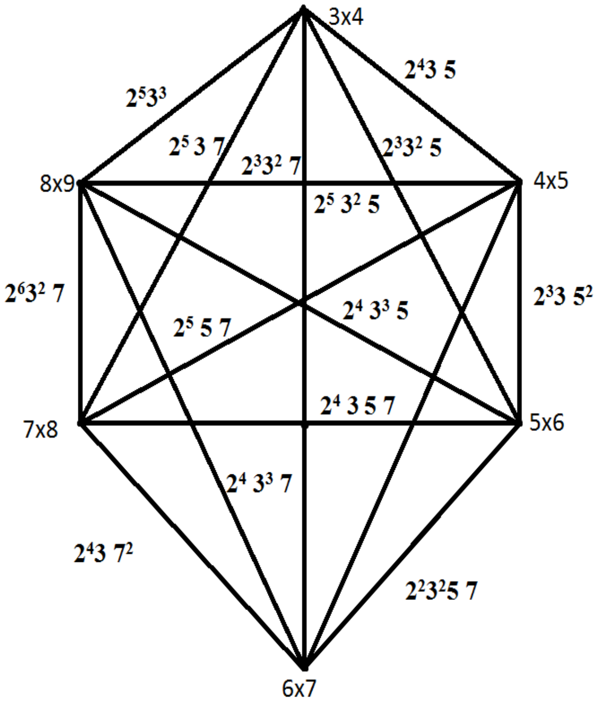

Fig.3.1. Zumkeller labelling of K_6

Definition 3.3

The one-point union of two complete graphs is a graph $K_n^{(2)}$ ($n \geq 3$) consists of two copies of K_n sharing a common vertex. It contains $2n-1$ vertices and $n(n-1)$ edges.

Theroem3.2:

The graph $K_n^{(2)}$ admits a Zumkeller labeling when $n \leq 6$.

Proof:

Let $V = \{v_i \mid 1 \leq i \leq n-1\} \cup \{u_i \mid 1 \leq i \leq n-1\} \cup \{v_0\}$ be the vertex set and

$$E = \{e_{ij} \mid 1 \leq i \leq n-1, 1 \leq j \leq n-1 \ and \ i < j\}$$
$$\cup \{e'_{ij} \mid 1 \leq i \leq n-1, 1 \leq j \leq n-1 \ and \ i < j\}$$
$$\cup \{e_{0i} \mid 1 \leq i \leq n-1\} \cup \{e_{0j} \mid 1 \leq j \leq n-1\}$$

be the edge set of the graph $K_n^{(2)}$.

Define an injective function $f: V \to N$ such that

$$f(v_0) = 8 \times 9$$
$$f(v_i) = (i+2) \times (i+3)$$
$$f(u_i) = (i+8) \times (i+9)$$

Define an induced function $f^*: E \to N$ such that

$$f^*(e_{ij}) = f^*(v_i v_j) = f(v_i) f(v_j)$$
$$f^*(e'_{ij}) = f^*(u_i u_j) = f(u_i) f(u_j)$$
$$f^*(e_{0i}) = f^*(v_0 v_i) = f(v_0) f(v_i) = 8 \times 9 f(v_i)$$
$$f^*(e_{0j}) = f^*(v_0 v_j) = f(v_0) f(v_j) = 8 \times 9 f(v_j)$$

All the edges receive the Zumkeller numbers. This can be proved as in Theorem3.1.

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Hence The graph $K_n^{(2)}$ admits a Zumkeller labeling when $n \le 6$.

Theorem 3.2 is illustrated in the following example 3.2.

Example 3.3

The Fig.3.2 shows the existence of the Zumkeller labeling for the graph $K_6^{(2)}$.

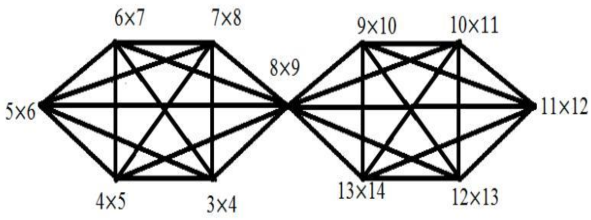

Fig.3.2. $K_6^{(2)}$ is a Zumkeller graph

Theorem 3.3

The fan graph $F_n = P_n + K_1$ is a Zumkeller graph.

Proof:

Let $V = \{v_0, v_1, v_2, \ldots, v_n, v_{n+1}\}$ be the vertex set of the fan graph F_n, where v_0 is the apex vertex.

Let

$$E = \{e_i = v_i v_{i+1} \mid 1 \le i \le n\} \cup \{e_i^{'} = v_0 v_i \mid 1 \le i \le n+1\}$$

be the edge set of the fan graph.

Define an injective mapping $f : V \to N$ such that

$$f(v_i) = 2^{\frac{i+1}{2}},$$

$$f(v_{i+1}) = p_1 2^{\frac{i+1}{2}}, \ i = 1, 3, 5 \ldots n$$

$f(v_0) = 2p_2$ where p_1, p_2 are distinct prime numbers which are greater than 2 but less than 10 and an induced function $f^* : E \to N$ such that

$$f^*(e_i) = f^*(v_i v_{i+1}) = f(v_i) f(v_{_{i+1}}), \ 1 \le i \le n$$

$$f^*(e_i^{'}) = f^*(v_0 v_i) = f(v_0) f(v_i), \ 1 \le i \le n+1$$

Now we claim that the numbers on the edges are Zumkeller numbers.

$$f^*(e_i) = f^*(v_i v_{i+1}) = f(v_i) f(v_{_{i+1}})$$

$$= 2^{\frac{i+1}{2}} p_1 2^{\frac{i+1}{2}} = p_1 2^{i+1}$$

is a Zumkeller number and

when $i \equiv 1 (\mathrm{mod}\ 2)$

(i) $f^*(e_i^{'}) = 2p_2 2^{\frac{i+1}{2}} = p_2 2^{\frac{i+3}{2}}$.

(ii) $f^*(e_{i+1}^{'}) = 2p_1 p_2 2^{\frac{i+1}{2}} = p_1 p_2 2^{\frac{i+3}{2}}$

are Zumkeller numbers.

Hence the fan graph F_n is a Zumkeller graph.

Example 3.3

The fan graph F_6 in Fig 3.3 is a Zumkeller graph when $p_1 = 3, p_2 = 5$

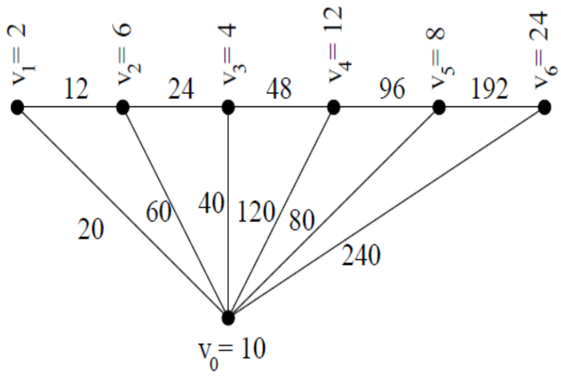

Fig 3.3. F_6 is a Zumkeller graph

IV. CONCLUSION

In this research article, the existence of the Zumkeller labeling to the following graphs

(i) The complete graph K_n, when $n \le 10$

(ii) The one-point union of two complete graphs when $n \le 6$.

(iii) The fan graph $F_n = P_n + K_1$

are proved.

Finding the other classes of graphs which will admit the Zumkeller labeling is a future scope of this research area.

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