# Bifurcation Analysis of Logistic Map Using Four Step Feedback Procedure

**Sudesh Kumari, Renu Chugh, Ashish Nandal**

***Abstract:*** ***The logistic map occupies a renowned place in the dynamics of chaos theory and in diverse areas of science. Picard orbit and Superior orbit (Mann orbit) have been used to control this discrete chaotic dynamical system. In this article, we further extend the analytical study of logistic map using a four step feedback procedure (SP orbit). The dynamical properties such as fixed point, range of convergence and stability, periodicity and chaos of the logistic map have been investigated. These properties are illustrated experimentally by adopting dynamical techniques like fixed point analysis and bifurcation plot. Using this approach, one can easily control the chaotic system and make the system stable for higher values of population growth parameter r by selecting the control parameters carefully.***

***Keywords*****:** ***Bifurcation plot, dynamical system, fixed point, four step feedback procedure, logistic map.***

## I. INTRODUCTION

The dynamical system is a system that exhibits sensitive dependence on the initial conditions, which means, the long term behavior of the dynamical system is unpredictable. Lorenz [1] and May [2] contributed a lot in the development of modern chaos theory. The logistic map is the simplest and well known one dimensional nonlinear dynamical system. It was primarily investigated by May [2] as a population growth model:

$$x_{n+1} = rx_n(1-x_n),\ n = 0,1,2,\dots,$$

where $r > 0$ is a nonlinear parameter and $x_n$ represents the value of $x$ after $n$ iterations. One may refer to Holmgren [3], Alligood et al. [4], Wiggins [5], Asuloos and Dirickx [6], Strogatz [7], Diamond [8], Robinson [9], Elhadj and Sprott [10] and Elagdi [11] to know more about the chaotic behavior of dynamical systems.

Nowadays, chaotic dynamical systems have been used for various purposes, for example, to modeling an image encryption in cryptography [12]-[14], to secure communication system and generate chaotic signals [15], to design noise generator system and QT circuit [16], [17] . Further, detailed scientific applications of logistic map have been presented by Peitgen et al. [18], Chowdhary et al. [19], Devaney [20], Chugh et al. [21], Sharkovsky et al. [22] etc.

Rani and Kumar [23], [24] studied logistic map via two step feedback procedures. Further, the capability of logistic map is enhanced by Rani and Goel [25] in I-Superior orbit. Recently, Kumari et al. [26] extended the capability of logistic map under SP orbit using time series representation. We shall study the dynamics of logistic map via SP orbit (a four step feedback procedure) which was investigated by W. Phuengrattana and S. Suantai [27] to prove that the SP-iteration converges faster than existing iterations.

In this article, the dynamics of logistic map including fixed point, chaos, range of convergence and stability is examined with the help of bifurcation plots. We show that the stability of logistic map can be increased drastically up to larger values of *r*, i.e., for $r = 36.38.$ The whole paper is split into four sections. In Section II, basic results, notations and definitions are included. Section III is dedicated to the experimental approach of logistic map and results obtained. Finally, in section IV, conclusion of the paper is presented.

## II. PRELIMINARIES

Now, we recall several definitions, facts and notations that are taken into account during our study.

**Definition 2.1** [20] The orbit of a point $z_0$ for a function $G$ is defined as the sequence of points $z_0,\ z_1 = G(z_0),\ z_2 = G^2(z_0), \dots,\ z_n = G^n(z_0), \dots$ .

**Definition 2.2** [20] Let $G: Z \to Z$ be a mapping where $Z$ is nonempty set. A point $z_0 \in Z$ is said to be fixed point of $G$ if $G(z_0) = z_0$ .

**Definition 2.3** [20] Let $G: Z \to Z$ be a mapping where $Z$ is a non-empty set. A point $z_0 \in Z$ is said to be periodic point of $G$ with period $p$ if $G^p(z_0) = z_0$ . The point $z_0$ has prime period $k$ if $G^k(z_0) = z_0$ and $G^n(z_0) \neq z_0$ for $0 < n < k$.

The orbit $O(G, z_0)$ is said to be periodic orbit of $z_0$ if $z_0$ is a periodic point.

**Definition 2.4** [18] The orbit $\{z_n\}$ of a point $z_0 \in Z$ defined by $O(G, z_0) = \{z_n : z_n = Gz_{n-1},\ n = 1, 2, \dots\}$, is known as Picard orbit, where the orbit $O(G, z_0)$ of a self map

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***** Correspondence Author

**Sudesh Kumari***, Department of Mathematics, Government College for Girls Sector 14, Gurugram-122001, India.

**Renu Chugh**, Department of Mathematics, Maharshi Dayanand University, Rohtak-124001, India.

**Ashish Nandal**, Department of Mathematics, Pt. NRS Government College, Rohtak-124001, India.

$G$ is the sequence $\{G^n(z_0)\}$ with initial point $z_0$.

**Definition 2.5** (SP Orbit) [27] Consider a sequence $\{x_n\}$ of iterates for initial point $x_0 \in Z$ such that

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n G y_n,$$
$$y_n = (1-\beta_n)z_n + \beta_n G z_n,$$
$$z_n = (1-\gamma_n)x_n + \gamma_n G x_n, \; n = 0,1,2,\ldots,$$

where $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ are sequences of positive numbers in [0,1]. This sequence of iterates is said to be SP orbit (four step feedback procedure). For the sake of simplicity, throughout the paper, we consider $\alpha_n = \alpha$, $\beta_n = \beta$ and $\gamma_n = \gamma$.

## III. EXPERIMENTAL ANALYSIS OF LOGISTIC MAP

This section deals with the bifurcation analysis of logistic map in SP orbit. Here, we try to find the maximum value of $r$ for which logistic map remains convergent and stable for all $x \in [0,1]$. Throughout the section, we analyze the dynamics of logistic map using SP orbit (four step feedback procedure). This entire experimental study is performed by running a program in Matlab.

Let $G_r(x) = rx(1-x)$ be the logistic map where $x \in [0,1]$ *and* $r > 0$. Using Definition 2.5, the logistic map in SP orbit can be expressed as

$$\begin{aligned} x_{n+1} &= (1-\alpha)y_n + \alpha G_r(y_n), \\ y_n &= (1-\beta)z_n + \beta G_r(z_n), \\ z_n &= (1-\gamma)x_n + \gamma G_r(x_n), \end{aligned} \qquad (1)$$

where $G_r(x_n) = rx_n(1-x_n)$; $\alpha, \beta, \gamma \in [0,1]$ and $n = 0,1,2,\ldots$.

Here, the dynamical behavior of the system (1) is controlled by three parameters $\alpha, \beta$ *and* $\gamma$. We study the behavior of logistic map by considering all these control parameters equal, i.e., $\alpha = \beta = \gamma$.

### A. Fixed Point Analysis of Logistic Map

The fixed points of the system (1) are computed by using Definition 2.2. But due to complexity of system (1), it is difficult to obtain the fixed points or solutions analytically. Therefore, to compute the fixed points, we run a program in Matlab and find out the following fixed points for different values of control parameters. The range of convergence and stability of the logistic map has also been computed.

**Table 1: Fixed points and range of convergence and stability of logistic map**

| $\alpha = \beta = \gamma$ | **Fixed point** | **Maximum value of $r$** | |
|---|---|---|---|
| | | **for convergence** | **for stability** |
| 0.9 | 0.6884 | 3.21 | 3.85 |
| 0.8 | 0.7134 | 3.49 | 4.21 |
| 0.7 | 0.7402 | 3.85 | 4.81 |
| 0.6 | 0.7685 | 4.32 | 5.24 |
| 0.5 | 0.7995 | 4.99 | 5.86 |
| 0.4 | 0.8330 | 5.99 | 6.66 |
| 0.3 | 0.8692 | 7.65 | 8.09 |
| 0.2 | 0.9089 | 10.98 | 11.24 |
| 0.1 | 0.9523 | 20.97 | 20.97 |
| 0.05 | 0.9725 | 36.38 | 36.38 |

From Table 1, we observe that the dynamical system (1) can be made stable for higher values of population growth parameter $r$, *i.e.*, for $r \le 36.38 \; \forall \, x \in [0,1]$. This extended range of convergent and stability of logistic map is obtained by decreasing control parameters $\alpha, \beta, \gamma \in [0,1]$.

### B. Bifurcation Analysis of Logistic Map

In this section, we adopt Bifurcation technique of dynamical systems to validate the results obtained in subsection A. The classification of dynamical systems in various regions can be done with the help of bifurcation diagrams. Here, we demonstrate that the whole dynamical behavior of logistic map (dynamical system) is controlled by the parameters $\alpha, \beta$ *and* $\gamma$ for some initial values of $x \in [0,1]$. The population growth parameter $r$ is taken on horizontal line and initial values $x_n$ are on vertical line.

The Fig. 1 represents the complete bifurcation plotting of logistic map for $\alpha = \beta = \gamma = 0.9$ and $1 \le r \le 4.5$. The whole dynamical system is divided into three regions (convergent, periodic and chaotic). From the figure, we observe that the orbit of logistic map converges to a fixed point and continues convergent till $r = 3.22$. The period doubling bifurcation of logistic map starts to occur for $3.22 < r \le 3.85$ as shown in Fig. 1. Further, logistic map becomes very sensitive on initial conditions, i.e., chaos starts to occur in the system when we increase $r$ through $r > 3.85$. In Fig. 2, this beauty of the chaos has been demonstrated by magnifying the bifurcation plot for $3.8 \le r \le 4.3$.

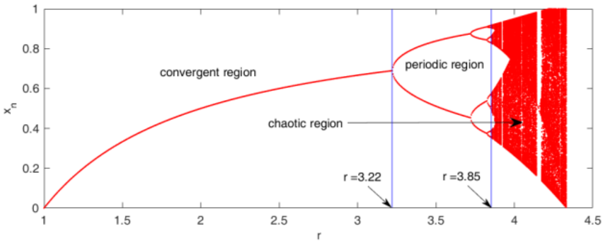


**Fig. 1 bifurcation plot of logistic map for**
$1 \le r \le 4.5,\ \alpha = \beta = \gamma = 0.9,\ x_0 = 0.5$

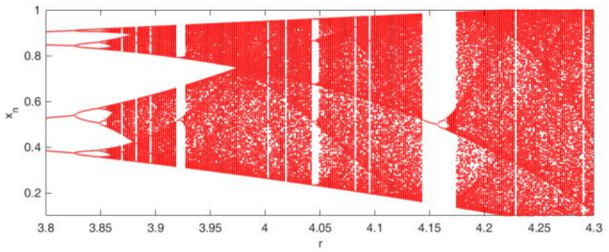


**Fig. 2 Magnified chaotic region of logistic map for**
$3.8 \le r \le 4.3,\ \alpha = \beta = \gamma = 0.9,\ x_0 = 0.5$

Furthermore, the period-doubling behavior of logistic map is presented in Fig. 3 by taking $\alpha = \beta = \gamma = 0.5$ for $4 \le r \le 7$. The map approaches to a fixed point, where $r$ varies from 0 to 4.99 and then it splits into two stable solutions expressed by periodic region for $4.99 < r \le 5.90$. The system cannot be described for $r > 5.90$ since the value of $x_n$ exceeds 1, i.e., $x_n \notin [0,1]$ which shows undefined logistic map. Thus, we can increase the stability and convergence range of dynamical system by decreasing the values of parameters $\alpha, \beta, \gamma$.

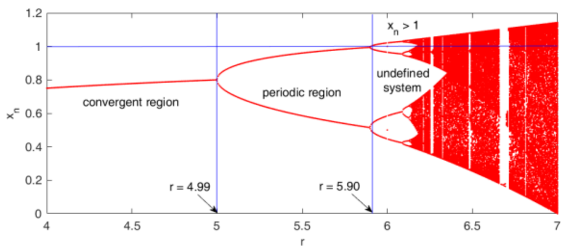


**Fig. 3 bifurcation plot of logistic map for**
$4 \le r \le 7,\ \alpha = \beta = \gamma = 0.5,\ x_0 = 0.3$

The entire bifurcation behavior of logistic map is also depicted for $\alpha = \beta = \gamma = 0.1$ in Fig. 4. For $\alpha = \beta = \gamma = 0.1$ and $r \le 20.97$, logistic map converges to a fixed point. But when the value of $r$ increases through $20.97$, logistic map cannot be described since $x_n$ is greater than 1. This undefined behavior of logistic map has been shown in Fig. 4.

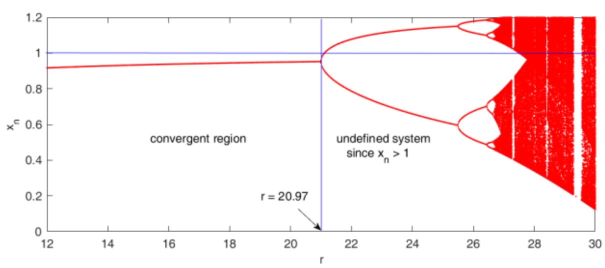


**Fig. 4 bifurcation plot of logistic map for**
$12 \le r \le 30,\ \alpha = \beta = \gamma = 0.1,\ x_0 = 0.9$

## IV. CONCLUSION

This bifurcation study of logistic map derives the results that (a) the logistic map exhibits all the dynamical properties i.e., fixed point, periodic points and chaos for $1 \le r \le 4.3$ at $\alpha = \beta = \gamma = 0.1$ (b) for $1 \le r \le 5.90$ and $\alpha = \beta = \gamma = 0.5$, logistic map exhibits fixed and periodic solutions but no chaos occurs (c) logistic map shows only convergent (fixed point) behavior for $1 \le r \le 20.97$ and $\alpha = \beta = \gamma = 0.1$. Thus, we conclude that as the values of control parameters $\alpha, \beta$ *and* $\gamma$ come closer to 0, logistic map loses dynamical properties like chaos and periodicity. Moreover, it is examined that it exhibits its stable and convergent behavior for a larger range of population growth parameter *r, i.e.*, for $0 < r \le 36.38$. This experimental analysis can be applied to control the chaotic behaviour of other dynamical systems also. Our further study is to design, implement and test an electronic circuit to accurately realize the logistic equation using this new four step feedback procedure.

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## AUTHORS PROFILE

**Ms Sudesh Kumari** is an Assistant Professor of Mathematics in Government College for Girls Sector 14, Gurugram. Also, she is pursuing Ph.D. in Mathematics from M.D. University, Rohtak. Several research papers published in various peer reviewed international journals are to her credit.

**Prof. Renu Chugh** is designated on the post of a Professor in Department of Mathematics, M. D. University, Rohtak and also, worked as head of the department from 2010 to 2013. Hon. President Shri Pranab Mukherjee has felicitated her on her selection as one of the 100 Women achievers of India on January 22, 2016. Also, Indian National Science Academy has awarded her with prestigious INSA Teachers Award 2017. She has 34 years research and teaching experience. She has supervised 27 Ph.D., 32 M.Phil. students and 2 Post doctoral fellows. She has published more than 185 research papers in reputed national and international journals.

**Mr Ashish Nandal** is an Assistant Professor in Department of Mathematics, Pt. NRS Govt. College, Rohtak. Also, he is pursuing his Ph.D. from Department of Mathematics, M.D. University, Rohtak. He has published several research papers in various peer reviewed international journals.