Fuzzy Paranormal Operators

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***Abstract:** In this paper, we introduced and discussed about fuzzy paranormal operators. A fuzzy bounded linear operator $\mathbb{T}$ on a fuzzy Hilbert space $\mathcal{H}$ is fuzzy paranormal if $\|\mathbb{T}^2 a\| \geq \|\mathbb{T}a\|^2$, for every unit vector a in $\mathcal{H}$. It is easily known that this class includes fuzzy hyponormal operators.*

***Keywords:** Fuzzy Hilbert space, Self-adjoint fuzzy operator, Fuzzy Normal operator, Fuzzy Hyponormal operator, Fuzzy Paranormal operator.*

I. INTRODUCTION

Let $\mathcal{H}$ be a fuzzy Hilbert space and FB $(\mathcal{H})$ is the set of all fuzzy bounded linear operators on $\mathcal{H}$. Biswas [10] first introduced the definition of fuzzy inner product space. In 2009, Goudarzi and Vaezpour [8] has been introduced the definition of a fuzzy Hilbert space.Sudad M Rasheed [4] was first introduced the concept and properties of adjoint fuzzy operator using the triplet $(\mathcal{H},\mathcal{F},*)$ and which is a fuzzy Hilbert space. An operator $\mathbb{T} \in FB(\mathcal{H})$ is a $\mathcal{T}_{\mathcal{F}}$ continuous linear functional, there exist $\mathbb{T}^* \in FB(\mathcal{H})$ such that$\langle \mathbb{T}a, b\rangle = \langle a, \mathbb{T}^*b\rangle\ \forall\ a, b \in \mathcal{H}$. Also $\mathbb{T}$ is a self – adjoint fuzzy operator if $\mathbb{T} = \mathbb{T}^*$ and also it commutes with its adjoint fuzzy operator i.e. $\mathbb{T}^*\mathbb{T} = \mathbb{T}\mathbb{T}^*$with this $\mathbb{T}$ is said to be fuzzy normal operator which was introduced by Radharamani et al. [1].

If $\mathbb{T} \in FB(\mathcal{H})$ is said to be fuzzy unitary operator if $\mathbb{T}^*\mathbb{T} = I = \mathbb{T}\mathbb{T}^*$.It is also a fuzzy isometry operator from $\mathcal{H}$ onto $\mathcal{H}$. In 2019, Fuzzy hyponormal operators and their properties are studied by Radharamani et al.[3] and investigated many interesting properties of Fuzzy hyponormal operators similar to these of fuzzy normal operators.An operator $\mathbb{T}$ is said to be fuzzy hyponormal if $\mathbb{T}^*\mathbb{T} \geq \mathbb{T}\mathbb{T}^*$.Also fuzzy class of N operators were defined if $\|\mathbb{T}^2 a\| \geq \|\mathbb{T}a\|^2\ \forall\ a \in \mathcal{H}, \|a\| = 1$.

Now we introduced fuzzy paranormal operator if $\|\mathbb{T}^2 a\|\|a\| \geq \|\mathbb{T}a\|^2\ \forall\ a \in \mathcal{H}$ which is equivalent to$\|\mathbb{T}^2 a\| \geq \|\mathbb{T}a\|^2$,for every unit vector a in $\mathcal{H}$. We have given an example, some lemmas for fuzzy paranormal operator and some properties like,sum and product of fuzzy paranormal operators are also fuzzy paranormal.An operator $\mathbb{T}$ is invertible and fuzzy paranormal, then $\mathbb{T}^{-1}$ also fuzzy paranormal.An operator $\mathbb{T}$ is fuzzy paranormal then its powers also fuzzy paranormal, also an operator $\mathbb{T}$ is fuzzy normal then $\mathbb{T}$ and $\mathbb{T}^*$ are also fuzzy paranormal.We will discuss these in detail.

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II. PRELIMINARIES

Definition 2.1: [9] Fuzzy Hilbert space (FH-space)

Let $(\mathcal{H},\mathcal{F},*)$be a FH – space with IP: $\langle a,b\rangle = Sup\,\{u \in R: \mathcal{F}(a,b,u) < 1\}\ \forall\ a,b \in \mathcal{H}$.If $\mathbb{T}$is complete in the $\|.\|$,then $\mathcal{H}$ is called Fuzzy Hilbert space(FH-space).

Definition 2.2: [4] Adjoint Fuzzy operator

Let $(\mathcal{H},\mathcal{F},*)$ be a FH – space and let $\mathbb{T} \in FB(\mathcal{H})$ be T_F continuous linear functional. Then $\exists$ unique $\mathbb{T}^* \in FB(\mathcal{H})$ such that $\langle \mathbb{T}a, b\rangle = \langle a, \mathbb{T}^*b\rangle \forall$ a, b $\in \mathcal{H}$.

Definition2.3: [4] Self-Adjoint Fuzzy operator

Let $(\mathcal{H},\mathcal{F},*)$ be a FH – space with IP: $\langle a,b\rangle = Sup\,\{u \in R: \mathcal{F}(a,b,u) < 1\}\ \forall\ a,b \in \mathcal{H}$ and let $\mathbb{T} \in FB(\mathcal{H})$ Then $\mathbb{T}$ is self-adjoint Fuzzy operator,if $\mathbb{T} = \mathbb{T}^*$,where $\mathbb{T}^*$ is adjoint Fuzzy operator of $\mathbb{T}$.

Theorem 2.4: [4]

Let $(\mathcal{H},\mathcal{F},*)$ be a FH – space with IP: $\langle a,b\rangle = Sup\,\{u \in R: \mathcal{F}(a,b,u) < 1\}\ \forall\ a,b \in \mathcal{H}$ and let $\mathbb{T} \in FB(\mathcal{H})$*then*,$\|\mathbb{T}a\| = \|\mathbb{T}^*a\|$for all $a, b \in \mathcal{H}$.

Theorem 2.5:[4]

Let $(\mathcal{H},\mathcal{F},*)$ be a FH – space with IP: $\langle a,b\rangle = Sup\,\{u \in R: \mathcal{F}(a,b,u) < 1\}\ \forall\ a,b \in \mathcal{H}$and let $\mathbb{T} \in FB(\mathcal{H})$, then $\mathbb{T}$ is self –adjoint Fuzzy operator.

Definition 2.6: [1] Fuzzy Normal operator

Let $(\mathcal{H},\mathcal{F},*)$be a FH – space with IP: $\langle a,b\rangle = Sup\,\{u \in R: \mathcal{F}(a,b,u) < 1\}\ \forall\ a,b \in \mathcal{H}$ and let $\mathbb{T} \in FB(\mathcal{H})$.then $\mathbb{T}$ is said to be an Fuzzy Normal operator if it commutes with its (fuzzy) adjoint i.e.$\mathbb{T}\mathbb{T}^* = \mathbb{T}^*\mathbb{T}$

Definition 2.7: [2] Fuzzy isometry operator

Let $(\mathcal{H},\mathcal{F},*)$ be an FH-space with IP: $\langle a,b\rangle = Sup\,\{u \in R: \mathcal{F}(a,b,u) < 1\}\ \forall\ a,b \in \mathcal{H}$and let an operator $\mathbb{T}$ on a Fuzzy Hilbert space $\mathcal{H}$.i.e., $\mathbb{T} \in FB(\mathcal{H})$.then $\mathbb{T}$ is said to be a Fuzzy isometry operator if $\|\mathbb{T}a\| = \|a\|$ for any $a \in \mathcal{H}$.i.e.,$\langle \mathbb{T}a, \mathbb{T}b\rangle = \langle a, b\rangle$.

Remark 2.8: [1]

Let $FB(\mathcal{H})$.the set of all fuzzy linear operators on $\mathcal{H}$.

Definition 2.8: [2] Fuzzy Unitary operator

Let $\mathbb{T} \in FB(\mathcal{H})$ is said to be a fuzzy unitary operator if $\mathbb{T}$ is a Fuzzy isometry operator from $\mathcal{H}$ onto $\mathcal{H}$.

Definition 2.9: [1] Fuzzy Hyponormal Operator

Let $(\mathcal{H},\mathcal{F},*)$ be a FH – space with IP: $\langle a,b\rangle = Sup\,\{u \in R: \mathcal{F}(a,b,u) < 1\}\ \forall\ a,b \in \mathcal{H}$ and let $\mathbb{T} \in FB(\mathcal{H})$Then $\mathbb{T}$ is a fuzzy hyponormal operator if $\|\mathbb{T}^*a\| \leq \|\mathbb{T}a\| \forall\ a \in \mathcal{H}$and or equivalently $\mathbb{T}^*\mathbb{T} - \mathbb{T}\mathbb{T}^* \geq 0$.

Theorem 2.11: [1]

Let $\mathbb{T} \in FB(\mathcal{H})$ be fuzzy hyponormal iff $\|\mathbb{T}^* a\| \leq \|\mathbb{T}a\|$ for all $a \in \mathcal{H}$.

III. MAIN RESULTS OF FUZZY PARANORMAL OPERATORS

Then $\mathbb{T}$ is a fuzzy paranormal operator if $\|\mathbb{T}^2 a\|\|a\| \geq \|\mathbb{T}a\|^2 \ \forall\, a \in \mathcal{H}$.

Note:

Let $\mathbb{T} \in FB(\mathcal{H})$. then $\mathbb{T}$ is a fuzzy paranormal operator if $\|\mathbb{T}^2 a\| \geq \|\mathbb{T}a\|^2$ for every unit vector a in $\mathcal{H}$.

Example 3.2:

Let $(\mathcal{H}, \mathcal{F}, *)$ be a fuzzy Hilbert space, $\mathcal{H} = l^2$. i.e. $l^2 = \{a = (a_1, a_2, a_3, \ldots) : \sum_{i=1}^{\infty} |a_i|^2 < \infty, a_i \in \mathbb{C}\}$ for a $\in l^2$, defined $\|a\| = \langle a, a\rangle^{\frac{1}{2}} = (\sum_{i=1}^{\infty} |a_i|^2)^{\frac{1}{2}}$ .Let $\mathcal{F}: l^2 \times (0, \infty) \to [0,1]$ define an operator $\mathbb{T}: l^2 \times l^2$ such that

$$\mathbb{T}(a_1, a_2, \ldots\ldots\ldots) = (0, a_1, a_2, \ldots\ldots\ldots)$$
$$\forall (a_1, a_2, \ldots\ldots\ldots) \in l^2$$

i)To find $\mathbb{T}$ is linear

Take $a = (a_1, a_2, \ldots\ldots\ldots)$, $b = (b_1, b_2, \ldots\ldots\ldots) \in l^2$ and scalar α.

$$\mathbb{T}(a + b) = \mathbb{T}(a_1 + b_1, a_2 + b_2 \ldots\ldots\ldots)$$
$$= (0, a_1 + b_1, a_2 + b_2 \ldots\ldots\ldots\ldots)$$
$$= (0, a_1, a_2, \ldots\ldots\ldots) + (0, b_1, b_2 \ldots\ldots\ldots)$$
$$\mathbb{T}(a + b) = \mathbb{T}(a) + \mathbb{T}(b)$$
$$\mathbb{T}(\alpha a) = (0, \alpha a_1, \alpha a_2, \ldots\ldots\ldots)$$
$$= \alpha(0, a_1, a_2, \ldots\ldots\ldots)$$
$$= \alpha\, \mathbb{T}(a)$$

ii) To find $\mathbb{T}$ is finite

Take $(a_1, a_2, \ldots\ldots\ldots) \in l^2$

$$\|\mathbb{T}(a_1, a_2, \ldots\ldots\ldots)\|^2 = \|(0, a_1, a_2, \ldots\ldots\ldots)\|^2$$
$$= \sum_{i=1}^{\infty} |a_i|^2$$
$$= \|a\|^2$$

i.e. $\|\mathbb{T}(a_1, a_2, \ldots\ldots\ldots)\|^2 = \|a\|^2$

$\|\mathbb{T}(a)\|^2 = \|a\|^2$ iff $\|\mathbb{T}(a)\| = \|a\|$

$\Rightarrow \mathbb{T}$ is finite

$$\therefore \mathbb{T} \in FB(l^2)$$

iii) To find $\mathbb{T}$ is fuzzy paranormal operator

Take $(a_1, a_2, \ldots\ldots\ldots) \in l^2$

$$\|\mathbb{T}(a_1, a_2, \ldots\ldots\ldots)\|^2 = \|(0, a_1, a_2, \ldots\ldots\ldots)\|^2$$
$$= \sum_{i=1}^{\infty} |a_i|^2$$
$$= \|(a_1, a_2, \ldots\ldots\ldots)\|^2$$
$$\Leftrightarrow \|\mathbb{T}(a_1, a_2, \ldots\ldots\ldots)\| = \|(a_1, a_2, \ldots\ldots\ldots)\|$$

iv) Take $(a_1, a_2, \ldots\ldots\ldots) \in l^2$

$$\|\mathbb{T}(a_1, a_2, \ldots\ldots\ldots)\| = \|(0, a_1, a_2, \ldots\ldots\ldots)\|$$

Let $\mathbb{T}^2(a_1, a_2, \ldots\ldots\ldots) = \mathbb{T}(\mathbb{T}(a_1, a_2, \ldots\ldots\ldots))$

$$= \mathbb{T}(0, a_1, a_2, \ldots\ldots\ldots)$$
$$\mathbb{T}^2(a_1, a_2, \ldots\ldots\ldots) = (0,0\ a_1, a_2, \ldots\ldots\ldots)$$
$$\|\mathbb{T}^2(a_1, a_2, \ldots\ldots\ldots)\| = \|(0,0\ a_1, a_2, \ldots\ldots\ldots)\|$$
$$\|\mathbb{T}^2(a_1, a_2, \ldots\ldots\ldots)\| = \sum_{i=1}^{\infty} |a_i|$$

v) Take any $(a_1, a_2, \ldots\ldots\ldots) \in l^2$

$$\mathbb{T}(a_1, a_2, \ldots\ldots\ldots) = (0, a_1, a_2, \ldots\ldots\ldots)$$

Definition 3.1:

Let $(\mathcal{H}, \mathcal{F}, *)$ be a FH – space with IP: $\langle a, b\rangle = Sup\,\{u \in R: \mathcal{F}(a, b, u) < 1\}\ \forall\, a, b \in \mathcal{H}$ and let $\mathbb{T} \in FB(\mathcal{H})$.

$$\|\mathbb{T}(a_1, a_2, \ldots\ldots\ldots)\|^2 = \|(0, a_1, a_2, \ldots\ldots\ldots)\|^2$$
$$= \sum_{i=1}^{\infty} |a_i|^2$$

From (iv) and (v),

$$\|\mathbb{T}^2(a)\| \geq \|\mathbb{T}(a)\|^2$$

Thus $\mathbb{T}$ is fuzzy paranormal operator.

Lemma 3.3:

Let $(\mathcal{H}, \mathcal{F}, *)$ be a FH – space with IP: $\langle a, b\rangle = Sup\,\{u \in R: \mathcal{F}(a, b, u) < 1\}\ \forall\, a, b \in \mathcal{H}$ and let $\mathbb{T} \in FB(\mathcal{H})$ be a fuzzy paranormal operator then $\|\mathbb{T}^3 a\| \geq \|\mathbb{T}^2 a\|\|\mathbb{T}a\|$ for every unit vector $a \in \mathcal{H}$.

Proof:

For a unit vector $a \in \mathcal{H}$,

Let $\|\mathbb{T}^3 a\|^2 = \langle \mathbb{T}^3 a, \mathbb{T}^3 a\rangle$

$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^3 a, \mathbb{T}^3 a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}\mathbb{T}^2 a, \mathbb{T}\mathbb{T}^2 a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^*\, \mathbb{T}\, \mathbb{T}^2 a, \mathbb{T}^2 a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^2\, \mathbb{T}^2 a, \mathbb{T}^2 a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^4 a, \mathbb{T}^2 a, u) < 1\}$$
$$= \langle \mathbb{T}^4 a, \mathbb{T}^2 a\rangle$$
$$\leq \|\mathbb{T}^4 a\|\|\mathbb{T}^2 a\|$$

$\|\mathbb{T}^3 a\|^2 \geq \|\mathbb{T}a\|^4 \|\mathbb{T}a\|^2$ [since $\mathbb{T}$ is fuzzy paranormal]

$\Rightarrow \|\mathbb{T}^3 a\| \geq \|\mathbb{T}a\|^2\, \|\mathbb{T}a\|$

Hence $\|\mathbb{T}^3 a\| \geq \|\mathbb{T}^2 a\|\|\mathbb{T}a\|$

Lemma 3.4:

Let $(\mathcal{H}, \mathcal{F}, *)$ be a FH – space with IP: $\langle a, b\rangle = Sup\,\{u \in R: \mathcal{F}(a, b, u) < 1\}\ \forall\, a, b \in \mathcal{H}$ and let $\mathbb{T} \in FB(\mathcal{H})$ be a fuzzy paranormal operator. Then $\|\mathbb{T}^{k+1} a\|^2 \geq \|\mathbb{T}^k a\|^2 \|\mathbb{T}^2 a\|$ for every positive integer $k \geq 1$ and every unit vector a in $\mathcal{H}$.

Proof:

Let $\mathbb{T} \in FB(\mathcal{H})$ be a fuzzy paranormal operator.By using the induction hypothesis, we will prove the theorem.

For the case $k = 1$,

$$\|\mathbb{T}^2 a\|^2 \geq \|\mathbb{T}a\|^2 \|\mathbb{T}^2 a\|$$

Now suppose that $\|\mathbb{T}^{k+1} a\|^2 \geq \|\mathbb{T}^k a\|^2 \|\mathbb{T}^2 a\|$ is valid for k.Then $k = k + 1$.

Let $\|\mathbb{T}^{k+2} a\|^2 = \langle \mathbb{T}^{k+2} a, \mathbb{T}^{k+2} a\rangle$

$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^{k+2} a, \mathbb{T}^{k+2} a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{T}^k)^* \mathbb{T}^{k+2} a, \mathbb{T}^2 a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{T}^*)^k \mathbb{T}^{k+2} a, \mathbb{T}^2 a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^{2k+2} a, \mathbb{T}^2 a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}\big(\mathbb{T}^{2(k+1)} a, \mathbb{T}^2 a, u\big) < 1\}$$
$$= \langle \mathbb{T}^{2(k+1)} a, \mathbb{T}^2 a\rangle$$
$$\leq \big\|\mathbb{T}^{2(k+1)} a\big\|\|\mathbb{T}^2 a\|$$

Since $\|\mathbb{T}^2 a\| \geq \|\mathbb{T}a\|^2 \|a\|\ \forall\, a \in \mathcal{H}$,

$$\|\mathbb{T}^{k+2} a\|^2 \geq \big\|\mathbb{T}^{(k+1)} a\big\|^2 \|\mathbb{T}^2 a\|$$

Hence the proof is obvious.

Lemma 3.5:

Let $\mathbb{T} \in FB(\mathcal{H})$ is a fuzzy paranormal operator.Then $\mathbb{T}^n$ is also fuzzy paranormal for every integer $n \geq 1$.

Proof:

It sufficient to prove that if $\mathbb{T}$ and $\mathbb{T}^k$ is a fuzzy paranormal then $\mathbb{T}^{k+1}$ is also fuzzy paranormal operator.

For every unit vector a in $\mathcal{H}$

Let$\left\|\mathbb{T}^{2(k+1)} a\right\|^2$=$\langle \mathbb{T}^{2(k+1)} a, \mathbb{T}^{2(k+1)} a \rangle$

$= Sup\,\{u \in R{:}\, \mathcal{F}\left(\mathbb{T}^{2(k+1)} a, \mathbb{T}^{2(k+1)} a, u\right) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}\left((\mathbb{T}^{2(k+1)})^* \mathbb{T}^{2(k+1)} a, a, u\right) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}\left((\mathbb{T}^*)^{2(k+1)} \mathbb{T}^{2(k+1)} a, a, u\right) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}(\mathbb{T}^{4k+4} a, a, u) < 1\}$

$= \langle \mathbb{T}^{4(k+1)} a, a \rangle$

$\leq \left\|\mathbb{T}^{4(k+1)} a\right\| \|a\|$

$\leq \left\|\mathbb{T}^{2(k+1)} a\right\| \left\|\mathbb{T}^{2(k+1)} a\right\| \|a\|$

i.e. $\left\|\mathbb{T}^{2(k+1)} a\right\|^2 \geq \left\|\mathbb{T}^{(k+1)}\right\|^4 \|a\|$

Implies that

$$\left\|\mathbb{T}^{2(k+1)} a\right\| \geq \left\|\mathbb{T}^{(k+1)} a\right\|^2$$

By the above lemma.

So $\mathbb{T}^{k+1}$ is also fuzzy paranormal operator.

Theorem 3.6:

Let $\mathbb{T} \in FB(\mathcal{H})$is a self-adjoint fuzzy operator then $\mathbb{T}$ is a fuzzy paranormal.

Proof:

For any a in $\mathcal{H}$ with $\|a\| = 1$, we know that $\mathbb{T}$ is a self-adjoint fuzzy operator. i.e.$\mathbb{T} = \mathbb{T}^*$.

Let $\|\mathbb{T}a\|^2 = \langle \mathbb{T}a, \mathbb{T}a \rangle$

$= Sup\,\{u \in R{:}\, \mathcal{F}(\mathbb{T}a, \mathbb{T}a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{T}^* \mathbb{T}a, a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{T}\mathbb{T}a, a, u) < 1\}$

$= \langle \mathbb{T}^2 a, a \rangle$

$\leq \|\mathbb{T}^2 a\| \|a\|$

$\|\mathbb{T}a\|^2 \leq \|\mathbb{T}^2 a\| \|a\|$

Implies that $\|\mathbb{T}a\|^2 \leq \|\mathbb{T}^2 a\|$

So $\mathbb{T}$ is a fuzzy paranormal operator.

Theorem 3.7:

Let $\mathbb{T} \in FB(\mathcal{H})$ be a fuzzy paranormal operator and self-adjoint fuzzy operator.Then $\mathbb{T}^*$ is fuzzy paranormal.

Proof:

For any $a \in \mathcal{H}, \|a\| = 1$

Let $\|\mathbb{T}^* a\|^2 = \langle \mathbb{T}^* a, \mathbb{T}^* a \rangle$

$= Sup\,\{u \in R{:}\, \mathcal{F}(\mathbb{T}^* a, \mathbb{T}^* a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{T}\mathbb{T}^* a, a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{T}^*)^2 a, a, u) < 1\}$

$= \langle (\mathbb{T}^*)^2 a, a \rangle$

$\leq \|(\mathbb{T}^*)^2 a\| \|a\|$

$\|\mathbb{T}^* a\|^2 \leq \|(\mathbb{T}^*)^2 a\| \|a\|$

Implies that $\|\mathbb{T}^* a\|^2 \leq \|(\mathbb{T}^*)^2 a\|$

i.e.$\|(\mathbb{T}^*)^2 a\| \geq \|\mathbb{T}^* a\|^2$

Therefore $\mathbb{T}^*$ is fuzzy paranormal.

Theorem 3.8:

Let $\mathbb{S}$ and $\mathbb{T} \in FB(\mathcal{H})$is a fuzzy paranormal operator and self-adjoint fuzzy operator.Then $\mathbb{S} + \mathbb{T}$ and $\mathbb{S}\mathbb{T}$ are also a fuzzy paranormal operator.

Proof:

For every unit vector a in $\mathcal{H}$, we know that $\|\mathbb{T}^2 a\| \geq \|\mathbb{T}a\|^2$,$\|\mathbb{S}^2 a\| \geq \|\mathbb{S}a\|^2$and $\mathbb{S} = \mathbb{S}^*$,$\mathbb{T} = \mathbb{T}^*$.

i).To prove that $\mathbb{S} + \mathbb{T}$ is a fuzzy paranormal operator.

Let $\|(\mathbb{S} + \mathbb{T})a\|^2 = \langle (\mathbb{S} + \mathbb{T})a, (\mathbb{S} + \mathbb{T})a \rangle$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{S} + \mathbb{T})a, (\mathbb{S} + \mathbb{T})a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{S} + \mathbb{T})^* (\mathbb{S} + \mathbb{T})a, a, u) < 1\}$

$= Sup\{u \in R{:}\, \mathcal{F}((\mathbb{S}^* + \mathbb{T}^*)(\mathbb{S} + \mathbb{T})a, a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}\left((\mathbb{S} + \mathbb{T})(\mathbb{S} + \mathbb{T})a, a, u\right) < 1\}$

$= \langle (\mathbb{S} + \mathbb{T})(\mathbb{S} + \mathbb{T})a, a \rangle$

$\leq \|(\mathbb{S} + \mathbb{T})^2 a\| \|a\|$

Implies that $\|(\mathbb{S} + \mathbb{T})a\|^2 \leq \|((\mathbb{S} + \mathbb{T})^2 a\|$

Therefore $\mathbb{S} + \mathbb{T}$ is a fuzzy paranormal operator.

ii). To prove that $\mathbb{S}\mathbb{T}$ is a fuzzy paranormal operator.

Let $\|(\mathbb{S}\mathbb{T})a\|^2$=$\langle (\mathbb{S}\mathbb{T})a, (\mathbb{S}\mathbb{T})a \rangle$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{S}\mathbb{T})a, (\mathbb{S}\mathbb{T})a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{S}\mathbb{T})^* (\mathbb{S}\mathbb{T})a, a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{T}\mathbb{S})(\mathbb{S}\mathbb{T})a, a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{S}\mathbb{T})(\mathbb{S}\mathbb{T})a, a, u) < 1\}$

$= \langle (\mathbb{S}\mathbb{T})(\mathbb{S}\mathbb{T})a, a \rangle$

$\leq \|((\mathbb{S}\mathbb{T})^2 a\| \|a\|$

$\|(\mathbb{S}\mathbb{T})a\|^2 \leq \|(\mathbb{S}\mathbb{T})^2 a\| \|a\|$

Implies that $\|(\mathbb{S}\mathbb{T})a\|^2 \leq \|((\mathbb{S}\mathbb{T})^2 a\|$

Therefore $\mathbb{S}\mathbb{T}$ is a fuzzy paranormal operator.

Theorem 3.9:

Let $\mathbb{T} \in FB(\mathcal{H})$ is a fuzzy normal operator. Then $\mathbb{T}$ is a fuzzy paranormal operator.

Proof:

For every unit vector a in $\mathcal{H}$

Let $\|\mathbb{T}a\|^2 = \langle \mathbb{T}a, \mathbb{T}a \rangle$

$= Sup\,\{u \in R{:}\, \mathcal{F}(\mathbb{T}a, \mathbb{T}a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{T}^* \mathbb{T}a, a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{T}\mathbb{T}^*)a, a, u) < 1\}$

$= \langle \mathbb{T}^2 a, a \rangle$

$\leq \|\mathbb{T}^2 a\| \|a\|$

i.e. $\|\mathbb{T}a\|^2 \leq \|\mathbb{T}^2 a\| \|a\|$

Implies that

$$\|\mathbb{T}a\|^2 \leq \|\mathbb{T}^2 a\|$$

Therefore$\mathbb{T}$ is fuzzy paranormal.

Theorem 3.10:

Let$\mathbb{T} \in FB(\mathcal{H})$ is a fuzzy paranormal operator and a fuzzy hyponormal operator.Then $\|\mathbb{T}\| \geq \|\mathbb{T}^*\|$ is a fuzzy paranormal operator.

Proof:

For every unit vector a in $\mathcal{H}$

Let $\|\mathbb{T}a\|^2 = \langle \mathbb{T}a, \mathbb{T}a \rangle$

$= Sup\,\{u \in R{:}\, \mathcal{F}(\mathbb{T}a, \mathbb{T}a, u) < 1\}$

$= Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{T}^* \mathbb{T}a, a, u) < 1\}$

$\geq Sup\,\{u \in R{:}\, \mathcal{F}((\mathbb{T}\mathbb{T}^*)a, a, u) < 1\}$

$\geq \langle \mathbb{T}^* a, \mathbb{T}^* a \rangle$

Since $\|\mathbb{T}^2 a\| \geq \|\mathbb{T}a\|^2$ and $\mathbb{T}^* \mathbb{T} - \mathbb{T}\mathbb{T}^* \geq 0 \;\forall\, a \in \mathcal{H}$,

$$\|\mathbb{T}a\|^2 \geq \|\mathbb{T}^* a\|^2$$
$$\Rightarrow \quad \|\mathbb{T}a\| \geq \|\mathbb{T}^* a\|$$
Implies that $\|\mathbb{T}\| \geq \|\mathbb{T}^*\|$.

Theorem 3.11:

Let $\mathbb{T}_n \in FB(\mathcal{H})$ is a sequence of fuzzy paranormal operator and $\mathbb{T}_n \to \mathbb{T}$ .Then $\mathbb{T}$ is a fuzzy paranormal operator.

Proof:

For every unit vector a in $\mathcal{H}$

Let $\|\mathbb{T}a\|^2 = \langle \mathbb{T}a, \mathbb{T}a\rangle$
$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}a, \mathbb{T}a, u) < 1\}$$
$$= \lim\, Sup\,\{u \in R: \mathcal{F}(\mathbb{T}_n a, \mathbb{T}_n a, u) < 1\}$$
$$= \lim\, Sup\,\{u \in R: \mathcal{F}((\mathbb{T}_n{}^* \mathbb{T}_n) a, a, u) < 1\}$$
$$= \lim \langle \mathbb{T}_n{}^* \mathbb{T}_n a, a\rangle$$
$$= \lim \langle \mathbb{T}_n{}^2 a, a\rangle$$
$$\|\mathbb{T}a\|^2 \leq \lim \left\|\mathbb{T}_n{}^2 a\right\| \|a\|$$
This implies $\|\mathbb{T}a\|^2 \leq \|\mathbb{T}^2 a\|$.
Hence $\mathbb{T}$ is a fuzzy paranormal operator.

Theorem 3.12:

Let $\mathbb{T} \in FB(\mathcal{H})$ is a fuzzy paranormal operator and $\mathbb{S}$ is unitarily equivalent to $\mathbb{T}$.Then $\mathbb{S}$ is a fuzzy paranormal operator.

Proof:

For $\mathbb{S}$ is unitarily equivalent to $\mathbb{T}$, we have $\mathbb{S} = \mathbb{U}\mathbb{T}\mathbb{U}^*$
For some unitarily equivalent to
$$\mathbb{S}^2 = \mathbb{U}\mathbb{T}^2\,\mathbb{U}^* \Rightarrow \|\mathbb{S}^2 a\| = \|\mathbb{U}\mathbb{T}^2\,\mathbb{U}^* a\|$$
Let $\|\mathbb{S}a\|^2 = \|(\mathbb{U}\mathbb{T}\mathbb{U}^*)a\|^2$
$$\langle \mathbb{S}a, \mathbb{S}a\rangle = \langle (\mathbb{U}\mathbb{T}\mathbb{U}^*)a, (\mathbb{U}\mathbb{T}\mathbb{U}^*)a\rangle$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{U}\mathbb{T}\mathbb{U}^*)a, (\mathbb{U}\mathbb{T}\mathbb{U}^*)a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{T}\mathbb{U}^*)a, \mathbb{U}^*\mathbb{U}(\mathbb{T}\mathbb{U}^*)a, u) < 1\}$$
$$= Sup\,\left\{u \in R: \mathcal{F}\left((\mathbb{T}\mathbb{U}^*)a, (\mathbb{T}\mathbb{U}^*)a, u\right) < 1\right\}$$
[$\because$ $\mathbb{U}$ is fuzzy isometry]
$$= Sup\,\{u \in R: \mathcal{F}(((\mathbb{T}\mathbb{U}^*)^*\mathbb{T}\mathbb{U}^*)a, a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{U}\mathbb{T}^*\mathbb{T}\mathbb{U}^*)a, a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{U}\mathbb{T}^2\,\mathbb{U}^*)a, a, u) < 1\}$$
$$\leq \|(\mathbb{U}\mathbb{T}^2\,\mathbb{U}^*)a\|\|a\|$$
$$\|\mathbb{S}a\|^2 \leq \|(\mathbb{U}\mathbb{T}^2\,\mathbb{U}^*)a\|\|a\|$$
Implies that
$$\|\mathbb{S}a\|^2 \leq \|\mathbb{S}^2 a\|\|a\|$$
Hence $\mathbb{S}$ is a fuzzy paranormal operator.

Theorem 3.13:

Let $\mathbb{T} \in FB(\mathcal{H})$ is an invertible and fuzzy paranormal operator. Then $\mathbb{T}^{-1}$ is also a fuzzy paranormal operator.

Proof:

For every unit vector a in $\mathcal{H}$

Let $\|\mathbb{T}a\|^2 = \langle \mathbb{T}a, \mathbb{T}a\rangle$
$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}a, \mathbb{T}a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{T}^*\mathbb{T}a, a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{T}\mathbb{T}^*)a, a, u) < 1\}$$
$$= \langle \mathbb{T}^2 a, a\rangle$$
$$\leq \|\mathbb{T}^2 a\|\|a\|$$
a is replaced by $\mathbb{T}^{-2} a$
$$\|\mathbb{T}\mathbb{T}^{-2} a\|^2 \leq \|\mathbb{T}^2\,\mathbb{T}^{-2} a\|\|\mathbb{T}^{-2} a\|$$
$$\|\mathbb{T}^{-1} a\|^2 \leq \|a\|\|\mathbb{T}^{-2} a\|$$
$$\|\mathbb{T}^{-1} a\|^2 \leq \|\mathbb{T}^{-2} a\|\|a\|$$
Implies that $\|\mathbb{T}^{-2} a\|\|a\| \geq \|\mathbb{T}^{-1} a\|^2$
i.e. $\|(\mathbb{T}^{-1})^2 a\|\|a\| \geq \|\mathbb{T}^{-1} a\|^2$
Hence $\mathbb{T}^{-1}$ is also afuzzy paranormal operator.

Theorem 3.14:

If $\mathbb{T}^{*^2}\mathbb{T}^2 \geq (\mathbb{T}^*\mathbb{T})^2$, then $\mathbb{T}$ is fuzzy paranormal operator.

Proof:

For every a in $\mathcal{H}$.
Let $\mathbb{T}^{*^2}\mathbb{T}^2 \geq (\mathbb{T}^*\mathbb{T})^2$
$$\mathbb{T}^{*^2}\mathbb{T}^2 - (\mathbb{T}^*\mathbb{T})^2 \geq 0$$
$$\langle (\mathbb{T}^{*^2}\mathbb{T}^2 - (\mathbb{T}^*\mathbb{T})^2)a, a\rangle \geq 0$$
$$Sup\,\left\{u \in R: \mathcal{F}\left((\mathbb{T}^{*^2}\mathbb{T}^2 - (\mathbb{T}^*\mathbb{T})^2)a, a, u\right) < 1\right\} \geq 0$$
$$Sup\,\left\{u \in R: \mathcal{F}\left((\mathbb{T}^{*^2}\mathbb{T}^2)a, a, u\right) < 1\right\} - Sup\,\{u \in R: \mathcal{F}((\mathbb{T}^*\mathbb{T})^2)a, a, u) < 1\} \geq 0$$
$$Sup\,\left\{u \in R: \mathcal{F}\left((\mathbb{T}^{*^2}\mathbb{T}^2)a, a, u\right) < 1\right\} \geq Sup\,\{u \in R: \mathcal{F}((\mathbb{T}^*\mathbb{T})^2)a, a, u) < 1\}$$
$$Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^2 a, \mathbb{T}^2 a, u) < 1\} \geq Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^*\mathbb{T}a, \mathbb{T}^*\mathbb{T}a, u) < 1\}$$
$\langle \mathbb{T}^2 a, \mathbb{T}^2 a\rangle \geq \langle \mathbb{T}^*\mathbb{T}a, \mathbb{T}^*\mathbb{T}a\rangle$ Since$\|\mathbb{T}^*\mathbb{T}\| = \|\mathbb{T}\|^2$
$$\|\mathbb{T}^2 a\|^2 \geq \|\mathbb{T}a\|^4$$
$$\Rightarrow \|\mathbb{T}^2 a\| \geq \|\mathbb{T}a\|^2$$
Hence $\mathbb{T}$ is fuzzy paranormal operator.

Theorem 3.15:

An operator $\mathbb{T} \in FB(\mathcal{H})$ is fuzzy paranormal if and only if $\mathbb{T}^{*^2}\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T} + k^2 \geq 0$ for all $k \in R$.

Proof:

For every$a$ in $\mathcal{H}$,
$$\mathbb{T}^{*^2}\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T} + k^2 \geq 0 \Leftrightarrow \langle (\mathbb{T}^{*^2}\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T} + k^2)\mathrm{u}, \mathrm{u}\rangle \geq 0$$
$$\mathbb{T}^{*^2}\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T} + k^2 \geq 0 \Leftrightarrow Sup\left\{u \in R: \mathcal{F}\left((\mathbb{T}^{*^2}\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T} + k^2)a, a, u\right) < 1\right\} \geq 0$$
$$\mathbb{T}^{*^2}\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T} + k^2 \geq 0 \Leftrightarrow Sup\,\left\{u \in R: \mathcal{F}\left((\mathbb{T}^{*^2}\mathbb{T}^2)a, a, u\right) < 1\right\} - 2kSup\,\left\{u \in R: \mathcal{F}\left((\mathbb{T}^*\mathbb{T})a, a, u\right) < 1\right\} + k^2\,Sup\,\{u \in R: \mathcal{F}(a, a, u) < 1\} \geq 0$$
$$\mathbb{T}^{*^2}\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T} + k^2 \geq 0 \Leftrightarrow Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^2 a, \mathbb{T}^2 a, u) < 1\} - 2kSup\,\{u \in R: \mathcal{F}(\mathbb{T}a, \mathbb{T}a, u) < 1\} + k^2\,Sup\,\{u \in R: \mathcal{F}(a, a, u) < 1\} \geq 0$$
$$\Leftrightarrow \langle \mathbb{T}^{*^2} a, \mathbb{T}^2 a\rangle - 2k\langle \mathbb{T}a, \mathbb{T}a\rangle + k^2\langle a, a\rangle \geq 0$$
$$\Leftrightarrow \|\mathbb{T}^2 a\|^2 - 2k\|Ta\|^2 + k^2\|a\|^2 \geq 0$$
Since if a > 0, b and c are real numbers then $at^2+bt+c \geq 0$ for every real t if and only if $b^2 - 4ac \leq 0$ in an analogous manner, using elementary property of real quadratic forms.
$$\Leftrightarrow 4\|Ta\|^4 - 4\|a\|^2\|\mathbb{T}^2 a\|^2 \leq 0$$
$$\Leftrightarrow \|Ta\|^4 \leq \|a\|^2\|\mathbb{T}^2 a\|^2$$
$$\Leftrightarrow \|Ta\|^2 \leq \|\mathbb{T}^2 a\|\|a\|$$
Therefore$\mathbb{T}$ is fuzzy paranormal operator.

Theorem 3.16:

Let $\mathbb{T} \in FB(\mathcal{H})$ is a fuzzy normal operator Then $\mathbb{T}^*$ is a fuzzy paranormal operator.

Proof:

Since $\mathbb{T}$ is a fuzzy normal operator.

We know that $\mathbb{T}^*\mathbb{T} = \mathbb{T}\mathbb{T}^*$ iff $\|\mathbb{T}^*a\| = \|\mathbb{T}a\|$

For every unit vector a in $\mathcal{H}$

Let $\|\mathbb{T}^*a\|^2 = \langle \mathbb{T}^*a, \mathbb{T}^*a\rangle$

$$= Sup\,\{u \in R: \mathcal{F}(\mathbb{T}^*a, \mathbb{T}^*a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{T}\mathbb{T}^*a, a, u) < 1\}$$
$$= Sup\,\{u \in R: \mathcal{F}((\mathbb{T}^*\mathbb{T})a, a, u) < 1\}$$
$$= \langle \mathbb{T}^{*2}a, a\rangle$$
$$\leq \|\mathbb{T}^{*2}a\|\|a\|$$
$$\|\mathbb{T}^*a\|^2 \leq \|\mathbb{T}^{*2}a\|\|a\|$$

Implies that

$$\|\mathbb{T}^*a\|^2 \leq \|\mathbb{T}^{*2}a\|$$

Therefore $\mathbb{T}^*$ is fuzzy paranormal.

Theorem 3.17:

Let $\mathbb{T} \in FB(\mathcal{H})$ is a fuzzy paranormal operator commutes with a fuzzy isometry operator $\mathbb{S}$. Then $\mathbb{T}\mathbb{S}$ is a fuzzy paranormal operator.

Proof:

Let $\mathcal{A} = \mathbb{T}\mathbb{S}$ for any real number k.

To prove $\mathcal{A}$ is a fuzzy paranormal operator.

That is to prove $\mathcal{A}^{*^2}\mathcal{A}^* - 2k\,\mathcal{A}^*\mathcal{A} + k^2 \geq 0$.

Now $\mathcal{A}^{*^2}\mathcal{A}^2 - 2k\,\mathcal{A}^*\mathcal{A} + k^2$

$$= (\mathbb{T}\mathbb{S})^{*^2}(\mathbb{T}\mathbb{S})^2 - 2k\,(\mathbb{T}\mathbb{S})^*(\mathbb{T}\mathbb{S}) + k^2$$
$$= (\mathbb{S}^*\mathbb{T}^*)^2(\mathbb{T}\mathbb{S})^2 - 2k\,(\mathbb{S}^*\mathbb{T}^*)(\mathbb{T}\mathbb{S}) + k^2$$
$$= \mathbb{T}^{*^2}(\mathbb{S}^{*^2}\mathbb{S}^2)\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T}(\mathbb{S}^*\mathbb{S}) + k^2$$
$$= \mathbb{T}^{*^2}(\mathbb{S}\mathbb{S}^*)^2\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T}(\mathbb{S}^*\mathbb{S}) + k^2$$

Since $\mathbb{T}$ is a fuzzy paranormal operator commutes with an fuzzy isometry operator $\mathbb{S}$.

$= \mathbb{T}^{*^2}\mathbb{T}^2 - 2k\mathbb{T}^*\mathbb{T} + k^2$ [by using theorem 2.15]

$$\mathcal{A}^{*^2}\mathcal{A}^2 - 2k\,\mathcal{A}^*\mathcal{A} + k^2 \geq 0$$
$$\Rightarrow (\mathbb{T}\mathbb{S})^{*^2}(\mathbb{T}\mathbb{S})^2 - 2k\,(\mathbb{T}\mathbb{S})^*(\mathbb{T}\mathbb{S}) + k^2 \geq 0$$

Hence $\mathbb{T}\mathbb{S}$ is a fuzzy paranormal operator.

IV. CONCLUSION

The conclusion that can be taken is a new idea of fuzzy paranormal operator in fuzzy Hilbert space, example and properties of fuzzy paranormal operators. Including its relationship with self-adjoint fuzzy operator, fuzzy normal operator and fuzzy hyponormal operators. In future, we hope it is very useful to find many types of fuzzy paranormal operators.

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