Certain Properties of Komatu Integral Transform with Negative Coefficients with Reference to Uniform Starlike and Uniform Convex Functions

R.Ambrose Prabhu, S.Bhaskaran, L.Vanitha, R.Ravi Kumar

ABSTRACT--- The authors obtained a new subclass about strongly starlike and strongly convex functions with respect to Komatu integral transforms and the inclusion properties of these classes such as \( S_\gamma T(\lambda, \gamma^n) \) and \( UCVT(\lambda, \gamma^n) \) were discussed. Furthermore, a new subclass about uniformly starlike functions along uniformly convex functions including negative coefficients defined by the Komatu integral transforms are introduced. The various properties about these classes are obtained here including (for instance) coefficient estimates, extreme points, distortion and covering theorems.

Mathematics Subject Classification: Primary 30C45

Keywords: Univalent function, Komatu integral transforms, starlike function and extreme points.

I. INTRODUCTION

The class of all analytic functions in the unit disk \( U = \{ z : |z| < 1 \} \) is symbolized as \( A \) and each \( \varphi \in A \), asserted to be

\[ \varphi(z) = z + \sum_{n=2}^\infty a_n z^n \quad (1) \]

Moreover, if each \( \varphi \in A \) satisfies the condition of univalent and normalization

i.e., \( \varphi(0) = 0 \) and \( \varphi'(0) = 1 \). Such a class is symbolized as \( \text{S} \).

Let \( T \) denotes the class that are analytic and univalent in \( U \), characterized in the pattern:

\[ \varphi(z) = z - \sum_{n=2}^\infty a_n z^n, \quad a_n \geq 0, \quad \forall \ n \geq 2, \quad (2) \]

Silverman[18] introduced the above class

Definition 1.1 When \( \varphi \in A \) convince the condition

\[ |a g \left( \frac{\omega\varphi(z)}{\varphi(z)} \right) - \gamma | < \frac{\pi}{2} \beta, \quad (3) \]

for few \( \gamma \) (\( 0 < \beta \leq 1 \)) and (\( 0 \leq \gamma < 1 \)), then \( \varphi(z) \) is termed as strongly starlike with order \( \beta \) and type \( \gamma \) in \( U \) and it is signified as \( S^*(\beta, \gamma) \).

Definition 1.2 When \( \varphi \in A \) convince the condition

\[ |a g \left( 1 + \frac{\omega\varphi(z)}{\varphi(z)} \right) - \gamma | < \frac{\pi}{2} \beta, \quad (4) \]

for few (\( 0 < \beta \leq 1 \)) and \( \gamma \) (\( 0 \leq \gamma < 1 \)), then \( \varphi(z) \) is termed as strongly convex of order \( \beta \) and type \( \gamma \) in \( U \) and it is signified as \( C(\beta, \gamma) \).

Definition 1.3 [5] A function \( \varphi \in S \) act as uniformly convex in \( U \) iff

\[ \Re \left\{ 1 + \frac{\varphi''(z)}{\varphi'(z)} \right\} \geq \frac{1}{\gamma}\varphi''(z) \text{,} \quad (z \in U) \]

and this class is symbolized as \( UCV \).

Rønning[16] and Ramachandran et al.[13,14] were introduced the following class of starlike functions analogous to \( UCV \) as follows:

Definition 1.4 A function \( \varphi \in S \) is termed as uniformly starlike in \( U \) iff

\[ \Re \left\{ \frac{\varphi'(z)}{\varphi(z)} \right\} \geq \frac{1}{\gamma}\varphi'(z) \text{,} \quad (z \in U) \]

and this class is symbolized as \( S^\gamma \).

In [15], Rønning also generalised the class \( S_p \) and is illustrated below:

Definition 1.5 A function \( \varphi \in S_p(\lambda), \ 0 \leq \lambda \leq 1, \) when \( \varphi \) convince the analytic characterization

\[ \Re \left\{ \frac{\varphi(z)}{\varphi(z)} \right\} - \lambda \geq \frac{\varphi'(z)}{\varphi'(z)} \text{,} \quad (z \in U) \]

and \( \varphi \in UCV(\lambda) \) iff \( \varphi \in S_p(\lambda) \).

The class \( UCV(\lambda) \) is defined in a similar manner(see[3]).

Definition 1.6 [8] The integral transform of \( \varphi \in T \) for \( \nu > -1, \rho > 0 \) is represented by \( T_\rho^\nu \varphi(z) \), is defined to be

\[ T_\rho^\nu \varphi(z) = \frac{(1+\rho)^\nu}{\Gamma(\rho)} \int_0^z (\log \frac{z}{t})^{\nu-1} \varphi(t) \text{dt} \]

\[ = z - \sum_{n=2}^\infty \left( \frac{1+\nu}{n+\nu} \right)^\rho a_n z^n, \quad (\nu > -1, \rho > 0), \quad (5) \]

is known as Komatu integral transform.

It was introduced and studied by Y. Komatu[8] and extended by the authors T. N.Shammugam and C.Ramachandran [17]. For \( \rho = 1 \), the generalized Bernardi-Libera-Livingston integral operator \( T_\rho^\nu \varphi(z) = L_\varphi(z) \) is given by

\[ L_\varphi(z) = \frac{(1+\nu)^\nu}{\Gamma(\nu)} \int_0^z \varphi(t) \text{dt} \]

\[ = z - \sum_{n=2}^\infty \left( \frac{1+\nu}{n+\nu} \right)^\nu a_n z^n, \quad (\nu > -1), \quad (6) \]

is studied by Bernadi[2] and the operator \( L_\varphi(z) \), for \( \nu = 1 \), \( L_1 \varphi(z) \) was investigated by Libera[9].
Certain Properties of Komatu Integral Transform with Negative Coefficients with Reference to Uniform Starlike and Uniform Convex Functions

For \( v = 1 \), Jung, Kim and Srivastava [7] introduced the one parameter family of integral operator \( T_v^p \phi(z) \) and also see [12]

\[
T_v^p \phi(z) = \frac{2^p}{z^2(p)_0} \int_0^2 (\log z)^p \phi(t)dt = z - \sum_{n=2}^{\infty} \left( \frac{2}{n+1} \right) a_n z^n, \quad (\rho > 0, f \in A).
\]

The operator \( T_v^p \phi(z) \) is considered before by Flett [4].

Using the belief of Bharathi et al.[3] and Jung et al.[7], consider the class \( S_pT(\lambda, l^v) \) and \( UCVT(\lambda, l^v) \) as follows:

**Definition 1.7** Let \( \phi \in S_pT(\lambda, l^v) \), \( 0 \leq \lambda < 1 \), arise the class of functions \( \phi \in T \) converge the condition

\[
\left| \frac{\phi(z_1)}{\phi(z)} - 1 \right| < 1 - \lambda, \quad z \in \mathbb{U}.
\]

**Definition 1.8** Let \( \phi \in UCVT(\lambda, l^v) \), \( 0 \leq \lambda < 1 \) arise the class of functions \( \phi \in T \) converge the condition

\[
\left| \frac{\phi(z_1)}{\phi(z)} - 1 \right| < 1 - \lambda, \quad z \in \mathbb{U}.
\]

**Lemma 1.9** If \( T_v^p \phi(z) \in T \), then

\[
\sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right) a_n \leq 1.
\]

**Proof.** Suppose

\[
\sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right) a_n > 1,
\]

there exits an integer \( N \) in such a way

\[
\sum_{n=1}^{N} \left( \frac{v+1}{v+n} \right) a_n > 1 + \epsilon/2
\]

\[
\left( \frac{1}{1+\epsilon/2} \right)^{1/2} < z < 1,
\]

we obtain

\[
(T_v^p \phi(z))' = 1 - \sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right) a_n z^{n-1}
\]

\[
\leq 1 - \sum_{n=2}^{N} \left( \frac{v+1}{v+n} \right) a_n z^{n-1}
\]

\[
\leq 1 - z^{N-1} \sum_{n=2}^{N} \left( \frac{v+1}{v+n} \right) a_n
\]

\[
\leq 1 - z^{N-1}(1 + \epsilon/2) < 0.
\]

But \( (T_v^p \phi(0))' = 1 > 0 \).

There exits a real number \( z_0 \), \( 0 < z_0 < 1 \), such that

\[
(T_v^p \phi(z_0)) = 0.
\]

Hence \( T_v^p \phi(z) \) is not univalent.

**Remark 1.10** For \( v = 1 \), Lemma 1.9 were discussed in [10].

By taking \( \rho = 1 \) Lemma 1.9, we can conclude the succeeding corollary:

**Corollary 1.11** Suppose \( \phi \in T \) be stated at (2). If \( l_v \in T \), then

\[
\sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right) a_n \leq 1.
\]

The prime objective is to investigate some coefficient estimates, distortion bounds, starlikeness and convexity for \( S_pT(\lambda, l^v) \) and \( UCVT(\lambda, l^v) \).

**II CHARACTERIZATION THEOREM & RESULTS**

Aqlan et al.[1] proposed a finest method to estimate the coefficient estimates for functions in classes \( S_pT(\lambda, l^v) \) and \( UCVT(\lambda, l^v) \). The main characterization theorem considering for the above classes are given as follows:

**Theorem 2.1** Suppose \( \phi \in T \). A function \( \phi \in S_pT(\lambda, l^v) \)

\[
\sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right)^p (2n - 1 - \lambda) a_n \leq 1 - \lambda, \quad (9)
\]

for some 0 \( \leq \lambda < 1 \) and \( \rho > 0 \). The result is sharp for

\[
\phi(z) = \sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right)^p (2n - 1 - \lambda) a_n z^n, \quad n \geq 2. \quad (10)
\]

**Proof.**

\[
\left| \frac{\phi(z_1)}{\phi(z)} - 1 \right| = \left| 1 - \frac{\phi(z_1)}{\phi(z)} \right| \leq \left| \frac{\phi(z_1)}{\phi(z)} \right| - 1 \leq \left| \frac{\phi(z_1)}{\phi(z)} \right| - 1 = 1 - \lambda
\]

which is equivalent to (7).

Conversely, if \( \phi \in S_pT(\lambda, l^v) \),

\[
1 - \sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right)^p a_n z^{n-1} < 0,
\]

\[
1 - \sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right)^p a_n z^{n-1} \leq \lambda
\]

\[
\geq 1 - \sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right)^p (n-1) a_n z^{n-1},
\]

\[
\sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right)^p a_n (2n - 1 - \lambda) \leq \lambda - 1
\]

Which is the expected result.

Since the proof of the Characterization theorem for the class \( UCVT(\lambda, l^v) \) is alike to the above theorem, it is skipped.

**Theorem 2.2** Suppose the function \( \phi \in T \). A function \( \phi \in UCVT(\lambda, l^v) \)

\[
\sum_{n=2}^{\infty} \left( \frac{v+1}{v+n} \right)^p (2n - 1 - \lambda) a_n \leq 1 - \lambda,
\]

for some 0 \( \leq \lambda < 1 \) and \( \rho > 0 \). The result is sharp for...
the function

\[ J_n^0 f(z) = z - \frac{1 - \lambda}{3 - \lambda} z^n, \quad n \geq 2. \]  

Equality holds true for \( J_n^0 f(z) \).

Equality holds true for \( J_n^0 f(z) \).

### III DISTORTION AND COVERING THEOREMS

**Theorem 3.1** If \( \varphi \in S_p^\mathcal{T}(\lambda, \lambda^0) \) and \( |z| = k < 1 \), the

\[ \frac{k - 1 - \lambda}{3 - \lambda} k^2 \leq \left| J_n^0 \varphi(z) \right| \leq \frac{k + 1 - \lambda}{3 - \lambda} k^2. \]

Equality holds true for \( J_n^0 \varphi(z) = z - \frac{1 - \lambda}{3 - \lambda} z^2 \).  

**Proof.** First, it is evident that

\[
(3 - \lambda) \sum_{n=2}^{\infty} \left( \frac{v + 1}{v + n} \right)^{\rho} a_n \leq \sum_{n=2}^{\infty} \left( \frac{v + 1}{v + n} \right)^{\rho} (2n - 1 - \lambda) a_n.
\]

If \( \varphi \in S_p^\mathcal{T}(\lambda, \lambda^0) \), using the inequality in Theorem 3.1,

\[
\sum_{n=2}^{\infty} \left( \frac{v + 1}{v + n} \right)^{\rho} a_n \leq \frac{1 - \lambda}{3 - \lambda} k^2.
\]

From (2) with \(|z| = k < 1\), we get

\[
\left| J_n^0 \varphi(z) \right| \leq k + \sum_{n=2}^{\infty} \left( \frac{v + 1}{v + n} \right)^{\rho} a_n k^n
\]

and

\[
\left| J_n^0 \varphi(z) \right| \geq k - \sum_{n=2}^{\infty} \left( \frac{v + 1}{v + n} \right)^{\rho} a_n k^n
\]

Equality holds true for \( J_n^0 \varphi(z) \).

Equality holds true for \( J_n^0 \varphi(z) \).

**Remark 3.3** For \( \nu = 1 \), Theorems 3.1 and 3.2 were discussed in [10].

By taking \( \rho = 1 \) in Theorems 3.1, 3.2, 3.3 and 3.4, we can conceive the successive corollaries:

**Corollary 2.6** Let the function \( \varphi \in T. A function \( \varphi \in S_p^\mathcal{T}(\lambda, \lambda^0) \) iff

\[
\sum_{n=2}^{\infty} \left( \frac{v + 1}{v + n} \right)^{\rho} (2n - 1 - \lambda) a_n \leq 1 - \lambda
\]

for some \( 0 \leq \lambda < 1 \). The result is sharp for \( J_n f(z) = z - \frac{1 - \lambda}{3 - \lambda} z^n, \ n \geq 2 \). (13)

**Corollary 2.7** Let the function \( \varphi \in T. A function \( \varphi \in UCVT(\lambda, \lambda^0) \) iff

\[
\sum_{n=2}^{\infty} \left( \frac{v + 1}{v + n} \right)^{\rho} (2n - 1 - \lambda) a_n \leq 1 - \lambda
\]

for some \( 0 \leq \lambda < 1 \). The result is sharp for \( J_n \varphi(z) = z - \frac{1 - \lambda}{3 - \lambda} z^n, \ n \geq 2 \). (15)

**Corollary 2.8** If \( \varphi \in S_p^\mathcal{T}(\lambda, \lambda^0) \), then

\[
a_n \leq \frac{1 - \lambda}{3 - \lambda} \left( \frac{v + 1}{v + n} \right)^{2n - 1 - \lambda}, \ n \geq 2.
\]

Equality holds true for the functions of the form given in (21).

**Corollary 2.9** Let the function \( \varphi \in T. A function \( \varphi \in UCVT(\lambda, \lambda^0) \), then

\[
a_n \leq \frac{1 - \lambda}{3 - \lambda} \left( \frac{v + 1}{v + n} \right)^{2n - 1 - \lambda}, \ n \geq 2.
\]

Equality holds true for the functions the form given in (23).
REFERENCES


8. Y. Komatu, On analytic prolongation of a family of operators, Math. (Cluj) 32 (1990), 199-211.


