International Journal of Engineering and Advanced Technology (IJEAT)
ISSN: 2249-8958 (Online), Volume-9 Issue-1, October, 2019

On the Strong Monophonic Number of a Graph

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Abstract : For a connected graph $G = (V, E)$ of order at least two, the strong monophonic problem is to determine a smallest set S of vertices of G such that, by fixing one monophonic path between each pair of the vertices of S, all vertices of G are covered. In this paper, certain general properties satisfied by the strong monophonic sets are studied. Also, the strong monophonic number of a several families of graphs and computational complexity are determined.

AMS Subject Classification: 05C12, 05C82

Keywords- monophonic set, monophonic number, strong monophonic set, strong monophonic number, monophonic distance.

I. INTRODUCTION

Geodetic and monophonic number of a graph are the widely studied variants in relation to the problem of the covering vertices of a graph with paths [1,2,4-8]. Over the recent years many variations of these concepts have been extensively studied in the literature.

For a non-trivial connected graph $G = (V(G), E(G))$, with vertices $u, v \in V(G)$, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. The eccentricity $e(u)$ of a vertex u is defined by $e(u) = max\{d(u, v) : v \in V\}$. The minimum and the maximum eccentricity among vertices of G is its radius r and diameter d, respectively. A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called monophonic if it is a chord less path. The monophonic distance $d_m(u, v)$ is defined as the length of a longest $u - v$ monophonic path in G. The monophonic eccentricity $e_m(v)$ of a vertex v in G is $e_m(v) = max\{d_m(u, v) : u \in V(G)\}$. The minimum and the maximum monophonic eccentricity among vertices of G is its monophonic radius r_m and monophonic diameter d_m, respectively. For other basic graph theoretic notation and terminology, we follow [3,9].

A set $S \subseteq V(G)$ is called a geodetic set if every vertex not in S lies on a geodesic between two vertices from S. The minimum cardinality of the geodetic set called the geodetic number, $g(G)$. If the monophonic paths are considered instead of shortest paths then we obtain the monophonic number of a graph. In other words, A set $S \subseteq V(G)$ of a graph G is a monophonic set of G if each vertex x of G lies on an $u - v$ monophonic path in G for some $u, v \in S$. The minimum cardinality of a monophonic set of G is the monophonic number, $m(G)$.

P. Manuel. et. al in [12] introduced a recent variation of geodetic set called the strong geodetic set. A set $S \subseteq V(G)$ is called a strong geodetic set if each vertex of G lies on the fixed geodesic between pairs of vertices from S. The minimum cardinality of strong geodetic set is called the strong geodetic number, denoted as $sg(G)$. It is shown that the problem of determining the strong geodetic number of a graph is an NP-complete problem. Many variants of monophonic set and monophonic number of a graph that are equivalent to geodetic concepts had been considered and studied in the literature [1, 2, 10, 11]. In this paper we formally define the strong monophonic number of a graph and study its properties.

II. STRONG MONOPHONIC NUMBER OF A GRAPH

Let $G = (V(G), E(G))$ be a non-trivial connected graph. If $S \subseteq V(G)$, then for each pair of vertices $u, v \in S, u \neq v$, let $\tilde{P}(u, v)$ be a **selected fixed monophonic path** between u and v. Then we set,

$$\tilde{J}(S) = \{\tilde{P}(u, v) : u, v \in S, u \neq v\} \text{ and}$$
$$V\left(\tilde{J}(S)\right) = \cup_{\tilde{P} \in \tilde{J}(S)} V(\tilde{P}).$$

If $V\left(\tilde{J}(S)\right) = V(G)$ for some $\tilde{J}(S)$, then the set S is called a **strong monophonic set.**

In other words, a set $S \subseteq V(G)$ is called a strong monophonic set if every vertex of G lies on a fixed monophonic path between pairs of vertices from S.

The **strong monophonic problem** is to find a minimum strong monophonic set S of G. The cardinality of the minimum strong monophonic set is the strong monophonic number and is denoted by $sm(G)$.

Example 2.1. For a theta graph G, the set $P = \{a, b\}$ forms a monophonic set giving $m(G) = 2$. But P is not a strong monophonic set for G. It can be easily verified that $S = \{c, d, e\}$ forms strong monophonic set, so $sm(G) = 3$. (Refer Fig 1:)

Revised Manuscript Received on October 30, 2019.
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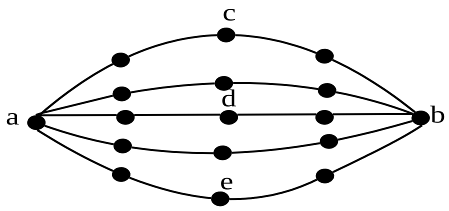

Fig 1.

III. COMPUTATIONAL COMPLEXITY

In this section, we prove that the strong monophonic number is NP-complete. The reduction is given from the problem of deciding whether given three vertices there is an induced path between two of the vertices passing through the third vertex.

Theorem 3.1. [13] Let x ; y ; z be three distinct vertices in a graph G. Deciding whether there is an induced path from x to y passing through z is NP-complete.

Using this result, we prove that deciding whether a subset set S of the vertex set of graph is a strong monophonic set or a strong open monophonic set is NP-complete.

Theorem 3.2. The strong monophonic problem for general graphs is NP-complete.

Proof: The proof is by reduction from the decision problem given in theorem 3.1. Let G be the given graph and suppose x ; y ; z be three distinct vertices. Construct G' as follows: Add two new vertices u and v such that they are adjacent to all vertices in $V(G)\backslash\{z\}$. Add an extreme vertex b to u. Similarly add extreme vertices $\{a_1, a_2, \dots, a_{n-3}\}$ to v. Let $V' = \{b, a_1, a_2, \dots, a_{n-3}\}$. Assume $S = V' \cup \{x, y\}$. Since the vertices of V' are the extreme vertices, it is contained in all strong monophonic sets. Clearly the paths between b and $\{a_1, a_2, \dots, a_{n-3}\}$ are unique fixed monophonic paths containing the vertices of $V(G)\backslash\{x, y, z\}$. Therefore z lies on the $x - y$ induced path if and only if S is a strong monophonic set of G'.

IV. MAIN RESULTS

Theorem 4.1. For any graph G,
$2 \le m(G) \le sm(G) \le n$

Proof: A monophonic set needs at least two vertices and therefore $m(G) \ge 2$. It is clear that every strong monophonic set is also a monophonic set and so $m(G) \le sm(G)$. Moreover, since the set of all vertices of G is a strong monophonic set of G, $sm(G) \le n$.

Remark 4.2: The bounds in theorem 4.1 are sharp. For the complete graph $K_n (n \ge 2)$, we have $sm(K_n) = n$ and for the path graph $P_n (n \ge 2)$, we have $sm(P_n) = 2$. The graphs with strong monophonic number equal to 2 are investigated in the sequel.

Theorem 4.3. Every strong monophonic set of a graph contains its extreme vertices.

Proof: Since every strong monophonic set is a monophonic set, the result follows.

Result 4.4 For any G, $sm(G) \le sg(G)$.
The bound is sharp for complete graphs and trees.

Corollary 4.5. For a graph G of order n with k extreme vertices, $max\{2, k\} \le sm(G) \le n$.

Proof: This follows from theorems 4.1 and 4.3.

Theorem 4.6. Let G be a connected graph with a cut vertex v. Then each strong monophonic set contains at least one vertex from each component of $G - v$.

Proof: This follows from the fact that every strong monophonic set is a monophonic set.

Theorem 4.7. Let T be a tree with l end-vertices, then $sm(T) = l$.

Proof: Since the end vertices are extreme, $l \le sm(T)$. For tree $sg(T) = l$ and by Result 4.4, $sm(T) \le sg(T) = l$. Therefore, $sm(T) = l$.

Corollary 4.8. For positive integers k, n, such that $2 \le k \le n$ there exist a connected graph G of order n , with $m(G) = sm(G) = k$.

Proof: For $k = n$, let $G = K_n$. Then, $m(G) = sm(G) = n = k$. Also, for each pair of integers k, n with $2 \le k < n$, there exists a tree of order n with k end vertices. Hence the result follows from theorem 4.7.

Theorem 4.9 For any connected graph G of order n, $sm(G) = n$ if and only if G is complete**.**

Proof: Let $sm(G) = n$. Suppose that G is not a complete graph. Then there exist two vertices u and v such that u and v are not adjacent in G. Since G is connected, there is a monophonic path from u to v, say P, with length at least 2. Let x be a vertex of P such that $x \neq u, v$. Then $S = V - \{x\}$ is a strong monophonic set of G and hence $sm(G) \le n - 1$, which is a contradiction. Thus G is a complete graph. Conversely if $G = K_n$, then clearly $sm(G) = n$.

Theorem 4. 10 Let G be a connected graph. Then $sm(G) = 2$ if and only if $G \cong P_n$.

Proof: If $G \cong P_n$, it is straight forward that $sm(G) = 2$. Now consider a graph $G(V, E)$ with $sm(G) = 2$. Let $u, v \in V(G)$ be any two non-adjacent vertices of G, such that all the other vertices of G lies on the $u - v$ monophoic path. Then it is clear that G is a path graph.

Theorem 4 11. If C_n be a cyle of order n, then $sm(C_n) = 3$.

Proof: By theorem 3.12, $sm(C_n) \ge 3$. Let $C_n: v_0, v_1, \dots, v_{n-1}, v_0$ and $S = \{v_0, v_1, v_2\}$. The set S is a strong monophonic set of C_n. Thus $sm(C_n) = 3$.

Theorem 4.12. For the grid $G_{n,m}$, $sm(G_{n,m}) = 3$, where integers $n, m \ge 3$.

Proof: By theorem 4.10, $sm(G_{n,m}) \ge 3$. Now it remains to show that there is a strong geodetic set S of $G_{n,m}$ of cardinality 3. Without loss of generality assume that $m \ge n$. Label each vertex of the grid $G_{n,m}$ as (x_i, y_j) if the vertex is in the i^{th} row and j^{th} coloumn, where $1 \le i \le n$ and $1 \le j \le m$. Let $S = \{(x_1, y_1), (x_1, y_m), (x_n, y_m)\}$.

Case[1] : Assume m is even.

Case [1(a)]: When $m \cong 0 \ mod \ 4$

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DOI: 10.35940/ijeat.A1231.109119
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All vertices of $G_{n,m}$ in the even coloumns except that of the m^{th} coloumn will lie on the $(x_1,y_1)-(x_n,y_m)$ fixed monophonic path, $P:(x_1,y_1),(x_1,y_2),(x_2,y_2)-(x_n,y_2)-(x_n,y_4),(x_{n-1},y_4)-(x_1,y_4),(x_1,y_5),(x_1,y_6)-(x_n,y_6)-(x_n,y_8),(x_{n-1},y_8),\dots\dots(x_1,y_{m-2})-(x_n,y_{m-2},)(x_n,y_{m-1})(x_n,y_m)$.

Similary, the vertices of the odd coloums in $G_{n,m}$ will lie on the $(x_1,y_1)-(x_1,y_m)$ fixed monophonic path $Q:(x_1,y_1),(x_2,y_1)-(x_n,y_1)-(x_n,y_3),(x_{n-1},y_3)-(x_1,y_3),(x_1,y_4),(x_1,y_5)-(x_n,y_5),\dots,(x_r,y_{m-1})-(x_1,y_{m-1}),(x_1,y_m)$.

Also the vertices in the m^{th} coloumn will lie on the $(x_1,y_m)-(x_n,y_m)$ fixed monophonic path.

Case [1(b)]: When $m \cong 2 \bmod 4$

All vertices of $G_{n,m}$ in the even coloumns except that of the m^{th} coloumn will lie on the $(x_1,y_1)-(x_1,y_m)$ fixed monophonic path, $R:(x_1,y_1),(x_1,y_2),(x_2,y_2)-(x_n,y_2)-(x_n,y_4),(x_{n-1},y_4)-(x_1,y_4),(x_1,y_5),(x_1,y_6)-(x_n,y_6)-(x_n,y_8),(x_{n-1},y_8),\dots\dots(x_n,y_{m-2})-(x_1,y_{m-2})-(x_1,y_m)$.

Similary, the vertices of the odd coloums in $G_{n,m}$ will lie on the $(x_1,y_1)-(x_n,y_m)$ fixed monophonic path $T:(x_1,y_1),(x_2,y_1)-(x_n,y_1)-(x_n,y_3),(x_{n-1},y_3)-(x_1,y_3),(x_1,y_4),(x_1,y_5)-(x_n,y_5),\dots,(x_1,y_{m-1})-(x_n,y_{m-1}),(x_n,y_m)$

Also the vertices in the m^{th} coloumn will lie on the $(x_1,y_m)-(x_n,y_m)$ fixed monophonic path.

Case[2] : Assume m is odd.

Case [2(a)]: When $m \cong 1 \bmod 4$

All vertices of $G_{n,m}$ in the even coloumns will lie on the $(x_1,y_1)-(x_1,y_m)$ fixed monophonic path, $U:(x_1,y_1),(x_1,y_2),(x_2,y_2)-(x_n,y_2)-(x_n,y_4),(x_{n-1},y_4)-(x_1,y_4),(x_1,y_5),(x_1,y_6)-(x_n,y_6)-(x_n,y_8),(x_{n-1},y_8),\dots\dots(x_n,y_{m-1})-(x_1,y_{m-1})(x_1,y_m)$

Similary, the vertices of the odd coloums in $G_{n,m}$ will lie on the $(x_1,y_1)-(x_n,y_m)$ fixed monophonic path $W:(x_1,y_1),(x_2,y_1)-(x_n,y_1)-(x_n,y_3),(x_{n-1},y_3)-(x_1,y_3),(x_1,y_4),(x_1,y_5)-(x_n,y_5),\dots,(x_1,y_{m-2}),(x_1,y_{m-1}),(x_1,y_m)-(x_n,y_m)$

Case [2(b)]: When $m \cong 3 \bmod 4$

All vertices of $G_{n,m}$ in the even coloumns will lie on the $(x_1,y_1)-(x_n,y_m)$ fixed monophonic path, $X:(x_1,y_1),(x_1,y_2),(x_2,y_2)-(x_n,y_2)-(x_n,y_4),(x_{n-1},y_4)-(x_1,y_4),(x_1,y_5),(x_1,y_6)-(x_n,y_6)-(x_n,y_8),(x_{n-1},y_8),\dots,(x_1,y_{m-1})-(x_n,y_{m-1}),(x_n,y_m)$.

Similary, the vertices of the odd coloums in $G_{n,m}$ will lie on the $(x_1,y_1)-(x_1,y_m)$ fixed monophonic path $Y:(x_1,y_1),(x_2,y_1)-(x_n,y_1)-(x_n,y_3),(x_{n-1},y_3)-(x_1,y_3),(x_1,y_4),(x_1,y_5)-(x_n,y_5),\dots,(x_n,y_{m-2}),(x_n,y_{m-1}),(x_n,y_m)-(x_1,y_m)$.

This implies $sm(G_{r,s})=3$.

Theorem 4.13 *Let G and H be connected graphs with $sm(G)=p$ and $sm(H)=q$, $2\le p\le q$. Then $sm(G\,\square H)\le pq-1$* and the bound is sharp.

Proof. Let $X = G\,\square\,H$. Let $S=\{g_1,\dots,g_p\}$ and $T=\{h_1,\dots,h_q\}$ be strong monophonic sets of G and H, respectively. Let $U=(S\times T)$. Suppose that the vertex $(x,y)\in X$, such that $(x,y)\notin U$. Since S is a strong monophonic set of G, say x lies on the fixed monophonic path $P_x:(g_i,g_{i'})$. Similarly since T is a strong monophonic set of H, say y lies on a fixed monophonic path $P_y:(h_j,h_{j'})$.

Case [1]: Assume $i\neq 1$ and $j\neq 1$. The vertex (x,y) will lie on the grid with adjacent sides as P_x and P_y and with corner vertices $\{(g_i,h_j),(g_i,h_{j'}),(g_{i'},h_j),(g_{i'},h_{j'})\}$. By Theorem 4.12, (x,y) lies on some fixed monophonic path between the pair of vertices of the set $\{(g_i,h_j),(g_i,h_{j'}),(g_{i'},h_j),(g_{i'},h_{j'})\}$.

Case [2]: Assume $i=1$ and $j=1$. Suppose $(g_i,h_j)=(g_1,h_1)$. By Theorem 4.12, (x,y) lie on the fixed monophonic path between some pair of these corner vertices of the set $\{(g_i,h_{j'}),(g_{i'},h_j),(g_{i'},h_{j'})\}$. Hence $U/\{x,y\}$ is a strong monophonic set and thus $sm(G\,\square H)\le pq-1$.

The bound is sharp for grids $(P_n\times P_m)$.

Theorem 4.14. For the n-dimensional hexagonal mesh $HX(n)$, $sm(HX(n))\le 4$, where $n\ge 2$.

Proof: By theorem 4.10, $sm(HX(n))\ge 3$. Let $S=\{\alpha,\beta,\gamma,\delta\}$ where α,β,γ and δ are vertices of $HX(n)$ shown in Fig. 2. Let $P_{\alpha,\beta}$ and $P_{\gamma,\delta}$ be two fixed monophonic paths between α,β and γ,δ, choosen as shown in Fig.2 (a) and (b) respectively. These paths cover all the vertices of $HX(n)$, implying S is a strong geodetic set of $HX(n)$.

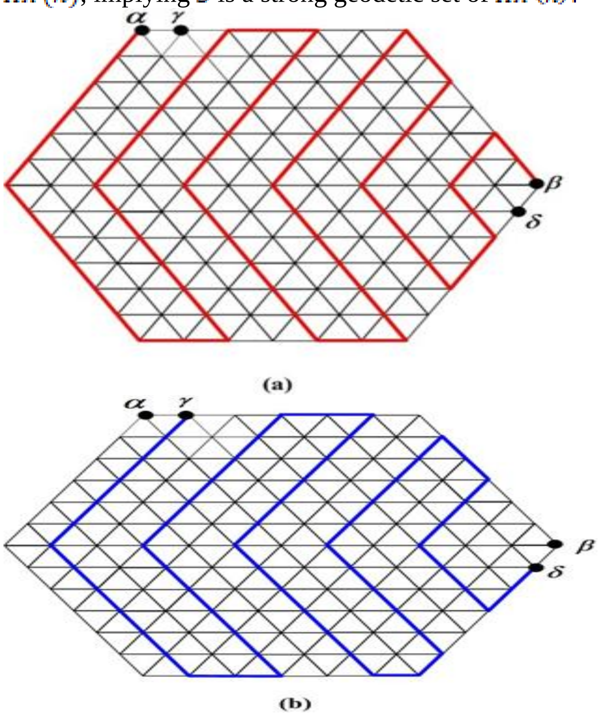

Fig. 2 A strong geodetic set of $HX(7)$

Theorem 4.15. Let $G = K_1 + \cup\, r_j\, K_j$ be a connected graph of order $n \geq 3$, where $\sum r_j \geq 2$. Then $m(G) = n - 1$.
Proof: Let $G = K_1 + \cup\, r_j\, K_j$, where $\sum r_j \geq 2$. Then G has only one cut vertex and all other $n - 1$ vertices are extreme vertices and hence by construction of G and by theorem 4.3, it is evident that $sm(G) = n - 1$.

Remark 4.16: The converse need not be true for consider the graph $K_n - e$, $sm(K_n - e) = n - 1$.

Theorem 4.17 For a n-dimensional hexagonal silicate network $SL(n)$ with k extreme vertices, $sm(G) = k$.
Proof: The simplicial vertices of $SL(n)$ are marked by white vertices in Fig. 3. It is easy to verify that the set of simplicial vertices forms a strong monophonic set. Hence the result follows from theorem 4.3.
In theorem 4.17, we consider only silicate networks of hexagonal type. But this result can be extended to any type of silicate sheets.

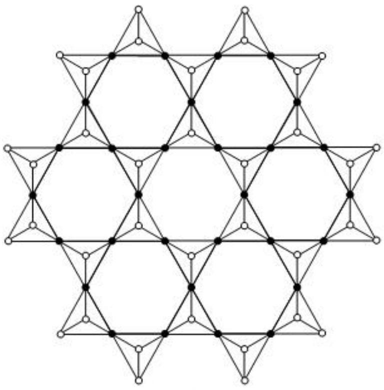

FIG 3

Theorem 4.18 For a graph $G(V, E)$ with $e \in E(G)$, $sm(G - e) \leq sm(G) + 2$
Proof: The edge e can either lie in some unique fixed monophonic path of G or it does not belong to any fixed monophonic paths in G.
Suppose the edge $e = uv \in E(G)$ lies in a unique monophonic path Q. With the removal of the edge $e = uv$, the monophonic path Q gets splitted to two other unique monophonic paths Q' and Q''. Thus the strong monophonic set of G togetherwith the vertices u and v gives a strong monophic set for $(G - e)$. Thus $sm(G - e) \leq sm(G) + 2$
Suppose the edge $e = uv \in E(G)$ does not belong to any fixed monophonic paths in G. Then the minimum strong monophonic set of G itself forms a strong monophonic set for $(G - e)$. Thus $sm(G - e) \leq sm(G)$. This implies that $sm(G - e) \leq sm(G) + 2$

Result 4.19 Let G be a Block graph, then the set of all extreme vertices form a minimum strong monophonic set for G.

Theorem 4.20 For complete bipartite graph $K_{m,n}$,

$$sm(K_{m,n}) = \begin{cases} 2\left\lceil \dfrac{-1+\sqrt{8m+1}}{2} \right\rceil & when\ 8m-1\ is\ not\ a\ perfect\ square \\ 2\left\lceil \dfrac{-1+\sqrt{8m+1}}{2} \right\rceil - 1 & when\ 8m-1\ is\ a\ perfect\ square \end{cases}$$

For complete bipartite graphs, the strong monophonic number coincides with the strong geodetic number and the result follows.

Result 4.21 For uniform theta-graph $\theta(l, n)$, the strong monophonic number,

$$sm(\theta(l, n)) = \left\lceil \frac{l}{2} \right\rceil$$

Theorem 4.22 For a split graph $G[K, T]$, where K is the complete set and T is the stable set with $|T| = a$, the $sm(G[K, T]) \geq \frac{n-b}{a}$, where b is the total number of extreme vertices in K.
Proof. For a split graph $G[K, T]$, the strong geodetic set and the strong monophonic set are equal. The bound for the minimum strong geodetic number for split graphs is given in [14] as $sg(G[K, T]) \geq \frac{n-b}{a}$ and the proof follows.

V. BOUNDS FOR STRONG MONOPHONIC NUMBER OF A GRAPH

Theorem 5.1 . If G is a non-trivial connected graph of order n and monophonic diameter d_m, then $sm(G) \leq p - d_m + 1$.
Proof. Let u and v be vertices of G such that $d_m(u, v) = d_m$ and let $P : u = v_0 v_1, \cdots, v_{d_m} = v$ be a $u - v$ monophonic path of length d_m. Let $S = V - \{v_1, v_2, \cdots, v_{d_m - 1}\}$. Clearly S is a strong monophonic set of G. Hence $sm(G) \leq p - d_m + 1$.

Theorem 5.2 If G is a non-trivial connected graph of order n and monophonic diameter $d_m \geq 2$ then, $sm(G) \geq \frac{d_m - 3 + \sqrt{(d_m - 3)^2 + 8n(d_m - 1)}}{2(d_m - 1)}$
Proof. Let S be the minimum strong monophonic set of G such that $|S| = sm(G)$. As the monophonic diameter of G is d_m, each monophonic path will have length at most d_m and thus covers at most $d_m - 1$ vertices of $G - S$. Moverover, since there are $sm(G)$ number of elements S, the given graph G is covered with $\binom{sm(G)}{2}$ monophonic paths. Therefore it follows that, $n \leq sm(G) + \binom{sm(G)}{2}(d_m - 1)$ which in turn implies, $(d_m - 1)sm(G)^2 - (d_m - 1)sm(G) - 2n \geq 0$. Since $sm(G)$ is a non-negative integer we conclude,

$$sm(G) \geq \frac{d_m - 3 + \sqrt{(d_m - 3)^2 + 8n(d_m - 1)}}{2(d_m - 1)}$$

Remark 5.3: The equality is obtained for the bound in Theorem 5.2 for path graphs. Further more we illustrate that the equality also holds for the grid graphs $G_{5,6}$.
Consider $G_{5,6}$. Then the number of vertices $n = 30$ and its monophonic diameter is $d_m = 20$. Hence by the bound in theorem 4.2, we have $sm(G_{5,6}) \geq 3$.
By Theorem 4.12 , $sm(G_{5,6}) = 3$.
The equality also holds for Petersen graph $GP(5,2)$, which stated in the theorem below.

Theorem 5.4 For a Petersen graph $GP(5,2)$, $sm(GP(5,2)) = 3$.
Proof. Let $GP(5,2)$ be the Petersen graph with vertex labelling as in Fig. 4.

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It follows from theorem 5.2 that $sm(GP(5,2)) \geq 3$. The set $S = \{u_1, u_2, u_4\}$ is a strong monophonic set. Hence $sm(GP(5,2)) = 3$.

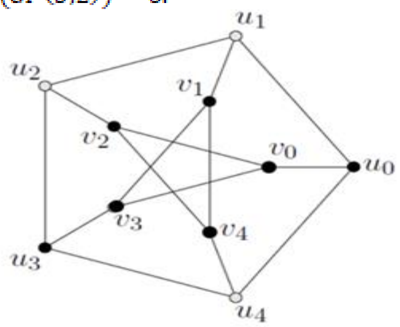

Fig. 4 A strong monophonic set of $GP(5,2)$

Theorem 5.5. For every pair a, b of positive integers with $2 \leq a \leq b$, there is a connected graph G with $m(G) = a$ and $sm(G) = b$.

Proof. For $2 \leq a = b$, any tree with a end vertices has $m(G) = sm(G) = a$. So, assume that $2 \leq a < b$. Let $P_i : x_i, w_i, y_i$ $(2 \leq i \leq a-1)$ be $a-2$ copies of a path of length 2. Similarly, $Q_i : m_i, z_i, n_i$ $(1 \leq i \leq b-a)$ be another set of $b-a$ copies of a path of length 2. Let $P : v_1, v_2, v_3, v_4$ a path of length 3. Now join each x_i $(2 \leq i \leq a-1)$ in P_i and v_2 in P , also join each m_i $(1 \leq i \leq b-a)$ in Q_i and v_2 in P and then join each y_i $(2 \leq i \leq a-1)$ in P_i and v_4 in P , also join each n_i $(1 \leq i \leq b-a)$ in Q_i and v_4 in P. Finally, add $a-1$ new vertices $u_1, u_2, \ldots, u_{a-1}$ and join each u_i $(1 \leq i \leq a-1)$ to v_4. The graph G is obtained. Let $S = \{v_1, u_1, u_2, \ldots, u_{a-1}\}$ be the set of all extreme vertices of G. It is easily verified that S is a monophonic set of G and so $m(G) = |S| = a$. But clearly this is not a strong monophonic set. Let $S' = S \cup \{z_1, z_2, \ldots, z_{b-a}\}$ and S' is the strong monophonic set. Moreover, it's easily verified that it is the minimum strong monophonic set. Hence $sm(G) = |S'| = b$.

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