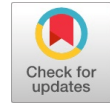


# The Forcing Restrained Steiner Number of a Graph



M. S. Malchijah Raj, J. John

**Abstract:** A restrained Steiner set of a connected graph  $G$  of order  $p \geq 2$  is a set  $W \subseteq V(G)$  such that  $W$  is a Steiner set, and if either  $W = V$  or the subgraph  $G[V - W]$  induced by  $V - W$  has no isolated vertices. The restrained Steiner number  $s_r(G)$  of  $G$  is the minimum cardinality of its restrained Steiner sets and any restrained Steiner set of cardinality  $s_r(G)$  is a minimum restrained Steiner set of  $G$ . For a minimum restrained Steiner set  $W$  of  $G$ , a subset  $T \subseteq W$  is called a forcing subset for  $W$  if  $W$  is the unique minimum restrained Steiner set containing  $T$ . A forcing subset for  $W$  of minimum cardinality is a minimum forcing subset of  $W$ . The forcing restrained Steiner number of  $W$ , denoted by  $f_{rs}(W)$ , is the cardinality of a minimum forcing subset of  $W$ . The forcing restrained Steiner number of  $G$ , denoted by  $f_{rs}(G)$  is  $f_{rs}(G) = \min \{f_{rs}(W)\}$ , where the minimum is taken over all minimum restrained Steiner sets  $W$  in  $G$ . Some general properties satisfied by the concept forcing restrained Steiner number are studied. The forcing restrained Steiner number of certain classes of graphs is determined. It is shown that for every pair  $a, b$  of integers with  $0 \leq a < b$  and  $b \geq 2$ , there exists a connected graph  $G$  such that  $f_{rs}(G) = a$  and  $s_r(G) = b$ .

**Keywords:**

Steiner distance, Steiner number, forcing Steiner number, restrained, Steiner number, forcing restrained Steiner number.

## I. INTRODUCTION

By a graph  $G = (V, E)$ , we mean a finite undirected graph without loops or multiple edges. The order and size of  $G$  are denoted by  $p$  and  $q$  respectively. The distance  $d(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of a shortest  $uv$  path in  $G$ . A  $u - v$  path of length  $d(u, v)$  is called an  $u - v$  geodesic. It is known that the distance is a metric on the vertex set  $V(G)$ . For basic graph theoretic terminology we refer to [6,1]. For a vertex  $v$  of  $G$ , the eccentricity  $e(v)$  is the distance between  $v$  and a vertex farthest from  $v$ . The minimum eccentricity among the vertices of  $G$  is the radius,  $radG$  and the maximum eccentricity is its diameter,  $diamG$  of  $G$ . Two vertices  $x$  and  $y$  are antipodal if  $d(x, y) = diamG$ . A vertex  $v$  is an extreme vertex of a graph  $G$  if the subgraph induced by its neighbors is complete. If  $e = \{u, v\}$  is an edge of a graph  $G$  with  $d(u) = 1$  and  $d(v) > 1$ , then we call  $e$  a pendant edge,  $u$  a leaf or end vertex and  $v$  a support. A geodetic set of  $G$  is a set  $S \subseteq V(G)$  such that every vertex of  $G$  is contained in a geodesic joining some pair of vertices in  $S$ . The geodetic number  $g(G)$  of  $G$  is the minimum cardinality of its geodetic sets and any geodetic set of cardinality  $g(G)$  is a minimum geodetic set.

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For a nonempty set  $W$  of vertices in a connected graph  $G$ , the Steiner distance  $d(W)$  of  $W$  is the minimum size of a connected subgraph of  $G$  containing  $W$ . Necessarily, each subgraph is a tree and is called a Steiner tree with respect to  $W$  or a Steiner  $W$  - tree. It is to be noted that  $d(W) = d(u, v)$ , when  $W = \{u, v\}$ . If  $v$  is an end vertex of a Steiner  $W$  - tree, then  $v \in W$ . Also if  $\langle W \rangle$  is connected, then any Steiner  $W$  - tree contains the elements of  $W$  only.  $S(W)$  denotes the set of all vertices that lie on Steiner  $W$  - trees. A set  $W \subseteq V(G)$  is called a Steiner set of  $G$  if every vertex of  $G$  lies on some Steiner  $W$  - tree or if  $S(W) = V(G)$ . A Steiner set of minimum cardinality is a minimum Steiner set or simply a  $s$  - set and this cardinality is the Steiner number  $s(G)$  of  $G$ . The Steiner number of a graph was introduced in [4]. The Steiner number of a graph was further studied in [5, 7, 11]. Steiner tree problem is a distance related invariant that arises by considering the cheapest subgraph that connects a given set of vertices. It has applications in design of computer circuits, long distance telephone lines, or mail routing, combinatorial optimization, etc. It is also used to construct roads of minimum total length to interconnect several highways. The computational nature of the problem makes it a traditional research subject in theory of computing. They have numerous application in industries as well. A set  $W$  of vertices of a graph  $G$  is a restrained Steiner set if  $W$  is a Steiner set, and if either  $W = V$  or the subgraph  $G[V - W]$  induced by  $V - W$  has no isolated vertices. The minimum cardinality of a restrained Steiner set of  $G$  is the restrained Steiner number of  $G$ , and is denoted by  $s_r(G)$ . A restrained Steiner set of minimum cardinality is called the  $s_r$  - set of  $G$ . This concept is introduced and studied in [8]. The forcing Steiner number of a graph is introduced in [13] and further studied in [14]. The forcing concept is applied in various graph parameters by several authors. In this paper we study the forcing concept in restrained Steiner set of a connected graph.

The following theorems are used in the sequel.

**Theorem 1.1.** [8] Each extreme vertex of a graph  $G$  belongs to every restrained Steiner set of  $G$ .

**Theorem 1.2.** [8] Let  $G$  be a non trivial tree which is not a star. Then  $s_r(G)$  is equal to the set of all end vertices of  $G$ .

**Theorem 1.3.** [8] For any tree  $T$  with  $p \geq 3$  vertices,  $s_r(T) = p$  if and only if  $T$  is a star.

# The Forcing Restrained Steiner Number of a Graph

Throughout the following  $G$  denotes a connected graph with at least two vertices.

## II. THE FORCING RESTRAINED STEINER NUMBER OF A GRAPH

Even though every connected graph contains a minimum restrained Steiner set, some connected graph may contain several minimum restrained Steiner sets. For each minimum restrained Steiner set  $W$  in a connected graph  $G$ , there is always some subset  $T$  of  $W$  that uniquely determines  $W$  as the minimum restrained Steiner set containing  $T$ . We consider such "forcing subsets" in this section.

**Definition 2.1** Let  $G$  be a connected graph and  $W$  a restrained Steiner set of  $G$ . A subset  $T \subseteq W$  is called a forcing subset for  $W$  if  $W$  is the unique minimum restrained Steiner set containing  $T$ . A forcing subset for  $W$  of minimum cardinality is a minimum forcing subset of  $W$ . The forcing restrained Steiner number of  $W$ , denoted by  $f_{rs}(W)$ , is the cardinality of a minimum forcing subset of  $W$ . The forcing restrained Steiner number of  $G$ , denoted by  $f_{rs}(G)$  is  $f_{rs}(G) = \min \{f_{rs}(W)\}$ , where the minimum is taken over all minimum restrained Steiner sets  $W$  in  $G$ .

**Example 2.2.** For the graph  $G$  given in Figure 2.1,  $W = \{v_1, v_3\}$  is the unique minimum restrained Steiner set of  $G$  so that  $f_{rs}(G) = 0$ . For the graph  $G$  given in Figure 2.2,  $W_1 = \{v_1, v_3, v_6\}$  and  $W_2 = \{v_1, v_4, v_6\}$  are the only two  $s_r$ -sets of  $G$ . It is clear that  $f_{rs}(W_1) = 1$  and  $f_{rs}(W_2) = 1$  so that  $f_{rs}(G) = 1$ .

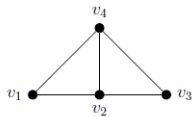


Figure 2.1

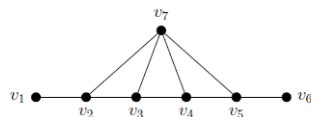


Figure 2.2

The next Theorem follows immediately from the definition of the restrained Steiner number and the forcing restrained Steiner number of a connected graph.

**Theorem 2.3.** For every connected graph  $G$ ,  $0 \leq f_{rs}(G) \leq s_r(G)$ .

**Remark 2.4.** The bounds in Theorem 2.3 are sharp. For the graph  $G$  given in Figure 2.1,  $f_{rs}(G) = 0$ . Also, all the inequalities in the theorem are strict. For the graph  $G$  given in Figure 2.2,  $f_{rs}(G) = 1$  and  $s_r(G) = 3$  so that  $0 < f_{rs}(G) < s_r(G)$ .

**Theorem 2.5.** Let  $G$  be a connected graph. Then

- $f_{rs}(G) = 0$  if and only if  $G$  has a unique minimum restrained Steiner set.
  - $f_{rs}(G) = 1$  if and only if  $G$  has at least two minimum restrained Steiner sets, one of which is a unique minimum restrained Steiner set containing one of its elements,
- and
- $f_{rs}(G) = s_r(G)$  if and only if no minimum restrained Steiner set of  $G$  is the unique

minimum restrained Steiner set containing any of its proper subsets.

**Definition 2.6.** A vertex  $v$  of a connected graph  $G$  is a restrained Steiner vertex of  $G$  if  $v$  belongs to every restrained Steiner set of  $G$ . If  $G$  has a unique restrained Steiner set  $W$ , then every vertex of  $W$  is a restrained Steiner vertex of  $G$ .

**Example 2.7.** For the graph  $G$  given in Figure 2.2,  $W_1 = \{v_1, v_3, v_6\}$  and  $W_2 = \{v_1, v_4, v_6\}$  are the only two  $s_r$ -sets of  $G$  so that every  $s_r$ -set contains the vertices  $v_1$  and  $v_6$ . Hence the vertices  $v_1$  and  $v_6$  are the restrained Steiner vertices of  $G$ .

**Theorem 2.8.** Let  $G$  be a connected graph and  $S$  be the set of all restrained Steiner vertices of  $G$ . Then  $f_{rs}(G) \leq s_r(G) - |S|$ .

*Proof.* Let  $W$  be a minimum restrained Steiner set of  $G$ . Then  $s_r(G) = |W|$ . It is clear that  $S \subseteq W$ . Let  $T$  be the unique minimum forcing restrained Steiner set containing  $W - S$ . Thus  $f_{rs}(G) \leq |W - S| \leq |W| - |S| = s_r(G) - |S|$ .

**Corollary 2.9.** If  $G$  is a connected graph with  $k$  extreme vertices, then  $f_{rs}(G) \leq s_r(G) - k$ .

*Proof.* The proof follows from Theorems 1.1 and 2.8.

**Remark 2.10.** The bound in Theorem 2.8 is sharp. For the graph  $G$  given in Figure 2.2,  $W_1 = \{v_1, v_3, v_6\}$  and  $W_2 = \{v_1, v_4, v_6\}$  are the only two  $s_r$ -sets of  $G$  such that  $f_{rs}(W_1) = 1$  and  $f_{rs}(W_2) = 1$  so that  $f_{rs}(G) = 1$  and  $s_r(G) = 3$ . Also,  $S = \{v_1, v_6\}$  is the set of all restrained Steiner vertices of  $G$  and so  $f_{rs}(G) = s_r(G) - |S|$ . Also, the inequality in Theorem 2.8 can be strict. For the graph  $G$  given in Figure 2.3,  $W_1 = \{v_1, v_4, v_7\}$  and  $W_2 = \{v_1, v_5, v_8\}$  are the only two  $s_r$ -sets of  $G$  so that  $s_r(G) = 3$ . It is clear that  $f_{rs}(W_1) = f_{rs}(W_2) = 1$  so that  $f_{rs}(G) = 1$ . Now,  $v_1$  is the only restrained Steiner vertex of  $G$  and so  $|S| = 1$ . Thus  $f_{rs}(G) < s_r(G) - |S|$ .

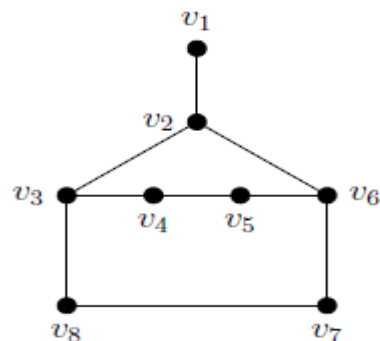


Figure 2.3

In the following we determine the forcing restrained Steiner number of some standard graphs.

**Theorem 2.11.** For the complete graph  $K_p$  ( $p \geq 3$ ),  $f_{rs}(K_p) = 0$ .

*Proof.* Since the set of all vertices of a complete graph  $K_p$  is the unique minimum restrained Steiner set, it follows from Theorem 2.5(a) that  $f_{rs}(K_p) = 0$ .

**Theorem 2.12.** For the complete bipartite graph  $K_{m,n}$  ( $m, n \geq 2$ ),  $f_{rs}(K_{m,n}) = 0$ .

*Proof.* For the complete bipartite graph  $K_{m,n}$  ( $m, n \geq 2$ ), we have  $s_r(K_{m,n}) = m + n$ . It follows from Theorem 2.5(a) that  $f_{rs}(K_{m,n}) = 0$ .

**Theorem 2.13.** Let  $G$  be a non trivial tree. Then  $f_{rs}(G) = 0$ .

*Proof.* If  $G$  is not a star, then by Theorem 1.2, the set of all end vertices of  $G$  is the unique minimum restrained Steiner set of  $G$ . Now, it follows from Theorem 2.5(a) that  $f_{rs}(G) = 0$ .

If  $G$  is a star  $K_{1,p-1}$ , then it follows from Theorem 1.3, that the set of all vertices of  $G$  is the unique minimum restrained Steiner set of  $G$ . By Theorem 2.5(a) it follows that  $f_{rs}(G) = 0$ .

**Theorem 2.14.** For the cycle  $C_p$  ( $p \geq 3$ ),  $f_{rs}(C_p) = \begin{cases} 0 & \text{for } p \in \{3, 4, 5\} \\ 1 & \text{for } p \geq 6 \text{ and } p \text{ is even.} \\ 2 & \text{for } p \geq 7 \text{ and } p \text{ is odd} \end{cases}$

*Proof.* It is proved in [8] that the restrained Steiner number  $s_r(C_p) = p$  for  $p \in \{3, 4, 5\}$  and so it follows from Theorem 2.5(a) that  $f_{rs}(C_p) = 0$  for  $p \in \{3, 4, 5\}$ . Let  $p \geq 6$ . For an even cycle  $C_p$ , the restrained Steiner set consists of a pair of antipodal vertices of  $C_p$  and from Theorem 2.5(b) it follows that  $f_{rs}(C_p) = 1$ . Let  $p$  be odd with  $p = 2n + 1, n = 1, 2, 3, \dots$

Let the cycle be  $C_p: v_1, v_2, \dots, v_n, v_{n+1}, v_{n+2}, \dots, v_{2n+1}, v_1$ . If  $W = \{u, v\}$  is any set of two vertices of  $C_p$ , then no vertex of the  $u - v$  longest path lies on the Steiner  $W$ -tree in  $C_p$  and so no two element subset of  $C_p$  is a Steiner set of  $C_p$ . Now it is clear that the sets  $W_1 = \{v_1, v_{n+1}, v_{n+2}\}, W_2 = \{v_2, v_{n+2}, v_{n+3}\}, \dots, W_{2n+1} = \{v_{2n+1}, v_n, v_{n+1}\}$  are the  $s_r$ -sets of  $C_p$ . Since  $p \geq 7$ , the subgraph  $G[V - W_i]$  ( $1 \leq i \leq 2n + 1$ ) has no isolated vertices. Therefore the  $s_r$ -sets of  $C_p$  are the  $s_r$ -sets of  $C_p$ . It is clear from the  $s_r$ -sets  $W_i$  ( $1 \leq i \leq 2n + 1$ ) that each  $\{v_i\}$  ( $1 \leq i \leq 2n + 1$ ) is a subset of more than one  $s_r$ -set  $W_i$ . Hence it follows from Theorem 2.5(a) and (b) that  $f_{rs}(C_p) \geq 2$ . Now since  $v_{n+1}$  and  $v_{n+2}$  are antipodal to  $v_1$ , it is clear that  $W_1$  is the unique  $s_r$ -set containing  $\{v_{n+1}, v_{n+2}\}$  and so  $f_{rs}(C_p) = 2$ . Hence the proof.

**Theorem 2.15.** For the wheel  $W_p = K_1 + C_{p-1}$ , ( $p \geq 5$ ),  $s_r(W_p) = p - 3$  and  $f_{rs}(W_p) = p - 4$ .

*Proof.* Let  $G = W_p$ . Let  $v$  be the vertex of  $K_1$  and let  $v_1, v_2, \dots, v_{p-1}, v_1$  be the cycle  $C_{p-1}$ . For  $p = 5$ , let  $W_1 = \{v_1, v_3\}$  and  $W_2 = \{v_2, v_4\}$ . Then  $W_i, i = 1, 2$  is a Steiner set of  $G$ . Since the subgraph  $G[V - W_i], i = 1, 2$  has no isolated vertices,  $W_i, i = 1, 2$  is a restrained Steiner set of  $G$  so that  $s_r(W_p) = 2 = p - 3$  and  $f_{rs}(W_p) = 1 = p - 4$ . Let  $p \geq 6$ . Let  $W$  be a Steiner set of  $G$ . If  $v \in W$ , then  $\langle W \rangle$  is connected. Then the Steiner  $W$ -tree contains elements of  $W$  only. Therefore  $v \notin W$ . Hence  $W \subseteq V(C_{p-1})$ . Let  $W$  be any subset of vertices of  $C_{p-1}$  of cardinality  $p - 3$  obtained by deleting two non-adjacent vertices of  $C_{p-1}$ . We may assume without loss of generality that  $W = \{v_1, v_2, \dots, v_{i-1}, v_{i+1}, v_{i+2}, \dots, v_{j-1}, v_{j+1}, v_{j+2}, \dots, v_{p-1}\}$ , where  $1 \leq i < j \leq p - 1$  and  $j \geq i + 2$ . Since the subgraph induced by  $G[V - W]$  has no isolated vertices, it is clear that  $W$  is a minimum restrained Steiner set of  $G$  so that  $s_r(W_p) \leq |W| = p - 3$ . Let  $W'$  be any restrained Steiner set with  $|W'| \leq p - 4$ . Then at least three vertices of  $C_{p-1}$  do not belong to  $W'$ . If these vertices are consecutive on  $C_{p-1}$ , then  $W'$  is not a Steiner set of  $G$ . Otherwise there are non adjacent vertices  $v_k$  and  $v_l$  ( $k < l$ ) such that  $v_k, v_l \notin W'$ . Since the size of any Steiner  $W'$ -tree is at most  $p - 4$ , it is easily seen that  $v_k$  and  $v_l$  do not lie on any Steiner  $W'$ -tree of  $G$  and so  $W'$  is not a restrained Steiner set of  $G$ . Thus  $s_r(W_p) = p - 3$ . Since the subgraph induced by a proper Steiner set is disconnected, it follows that any  $s_r$ -set is of the

form  $W = \{v_1, v_2, \dots, v_{i-1}, v_{i+1}, v_{i+2}, \dots, v_{j-1}, v_{j+1}, v_{j+2}, \dots, v_{p-1}\}$ , where  $v_i$  and  $v_j$  are non-adjacent. Let  $T$  be a subset of  $W$  with  $|T| \leq p - 5$ . Since  $p \geq 6$ , there exists distinct vertices  $x, y \in W$  such that  $x, y \notin T$ . If  $x$  and  $y$  are adjacent, then  $x$  is non-adjacent to at least one of  $v_i$  and  $v_j$ , say  $v_j$ . Then  $W_1 = V(C_{p-1}) - \{x, v_j\}$  is a  $s_r$ -set such that  $W_1 \neq W$  and  $W_1$  properly contains  $T$ . If  $x$  and  $y$  are non-adjacent, then  $W_2 = V(C_{p-1}) - \{x, y\}$  is a  $s_r$ -set such that  $W_2 \neq W$  and  $W_2$  properly contains  $T$ . Thus  $T$  is not a forcing subset for  $W$ . Now, we show that there exists a forcing subset of  $W$  of cardinality  $p - 4$ . For convenience, let  $W = \{v_2, v_4, v_5, v_6, \dots, v_{p-1}\}$ . We show that  $T_1 = \{v_4, v_5, v_6, \dots, v_{p-1}\}$  is a forcing subset of  $W$ . If  $T_1$  is not a forcing subset for  $W$ , then there exists a  $s_r$ -set  $W' \neq W$  such that  $T_1 \subseteq W'$ . Since  $W' \neq W, |W'| = p - 3$  and  $|T_1| = p - 4, W'$  must contain exactly one of  $v_1$  or  $v_3$ . In any case,  $\langle W' \rangle$  is connected and so  $W'$  is not a restrained Steiner set of  $G$ , which is a contradiction. Hence  $f_{rs}(W_p) = p - 4$ .

**Theorem 2.16.** If  $W = \{u, v\}$  is a  $s_r$ -set of a connected graph  $G$ , then  $u$  and  $v$  are two antipodal vertices of  $G$ .

*Proof.* Let  $W = \{u, v\}$  be a  $s_r$ -set of  $G$ . Then every vertex of  $G$  lies on a Steiner  $W$ -tree of  $G$ . Since every Steiner  $W$ -tree is a  $u - v$  geodesic, every vertex of  $G$  lies on a geodesic joining  $u$  and  $v$ . Also the subgraph  $G[V - W]$  has no isolated vertices. We claim that  $d(u, v) = d(G)$ . If  $d(u, v) < d(G)$ , then let  $x$  and  $y$  be two vertices of  $G$  such that  $d(x, y) = d(G)$ . Now, it follows that  $x$  and  $y$  lie on distinct geodesics joining  $u$  and  $v$ .



