The Forcing Restrained Steiner Number of a Graph

M. S. Malchijah Raj, J. John

Abstract: ***A restrained Steiner set of a connected graph G of order $p \geq 2$ is a set $W \subseteq V(G)$ such that W is a Steiner set, and if either $W = V$ or the subgraph $G[V - W]$ induced by $[V - W]$ has no isolated vertices. The restrained Steiner number $s_r(G)$ of G is the minimum cardinality of its restrained Steiner sets and any restrained Steiner set of cardinality $s_r(G)$ is a minimum restrained Steiner set of G. For a minimum restrained Steiner set W of G, a subset $T \subseteq W$ is called a forcing subset for W if W is the unique minimum restrained Steiner set containing T. A forcing subset for W of minimum cardinality is a minimum forcing subset of W. The forcing restrained Steiner number of W, denoted by $f_{rs}(W)$, is the cardinality of a minimum forcing subset of W. The forcing restrained Steiner number of G, denoted by $f_{rs}(G)$ is $f_{rs}(G) = \min\{f_{rs}(W)\}$, where the minimum is taken over all minimum restrained Steiner sets W in G. Some general properties satisfied by the concept forcing restrained Steiner number are studied. The forcing restrained Steiner number of certain classes of graphs is determined. It is shown that for every pair a, b of integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G such that $f_{rs}(G) = a$ and $s_r(G) = b$.***

Keywords:
Steiner distance, Steiner number, forcing Steiner number, restrained, Steiner number, forcing restrained Steiner number.

I. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest uv path in G. A $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. It is known that the distance is a metric on the vertex set $V(G)$. For basic graph theoretic terminology we refer to [6,1]. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, $rad\,G$ and the maximum eccentricity is its diameter, $diam\,G$ of G. Two vertices x and y are *antipodal* if $d(x, y) = diam\,G$. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. If $e = \{u, v\}$ is an edge of a graph G with $d(u) = 1$ and $d(v) > 1$, then we call e a pendant edge, u a leaf or end vertex and v a support. A geodetic set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $g(G)$ of G is the minimum cardinality of its geodetic sets and any geodetic set of cardinality $g(G)$ is a minimum geodetic set.

For a nonempty set W of vertices in a connected graph G, the Steiner distance $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W - tree. It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. If v is an end vertex of a Steiner W - tree, then $v \in W$. Also if $\langle W \rangle$ is connected, then any Steiner W - tree contains the elements of W only. $S(W)$ denotes the set of all vertices that lie on Steiner W - trees. A set $W \subseteq V(G)$ is called a Steiner set of G if every vertex of G lies on some Steiner W - tree or if $S(W) = V(G)$. A Steiner set of minimum cardinality is a minimum Steiner set or simply a s - set and this cardinality is the Steiner number $s(G)$ of G. The Steiner number of a graph was introduced in [4]. The Steiner number of a graph was further studied in [5, 7, 11]. Steiner tree problem is a distance related invariant that arises by considering the cheapest subgraph that connects a given set of vertices. It has applications in design of computer circuits, long distance telephone lines, or mail routing, combinatorial optimization, etc. It is also used to construct roads of minimum total length to interconnect several highways. The computational nature of the problem makes it a traditional research subject in theory of computing. They have numerous application in industries as well. A set W of vertices of a graph G is a restrained Steiner set if W is a Steiner set, and if either $W = V$ or the subgraph $G[V - W]$ induced by $V - W$ has no isolated vertices. The minimum cardinality of a restrained Steiner set of G is the restrained Steiner number of G, and is denoted by $s_r(G)$. A restrained Steiner set of minimum cardinality is called the s_r - set of G. This concept is introduced and studied in [8]. The forcing Steiner number of a graph is introduced in [13] and further studied in [14]. The forcing concept is applied in various graph parameters by several authors. In this paper we study the forcing concept in restrained Steiner set of a connected graph.

The following theorems are used in the sequel.

Theorem 1.1. [8] *Each extreme vertex of a graph G belongs to every restrained Steiner set of G.*

Theorem 1.2. [8] *Let G be a non trivial tree which is not a star. Then $s_r(G)$ is equal to the set of all end vertices of G.*

Theorem 1.3. [8] *For any tree T with $p \geq 3$ vertices, $s_r(T) = p$ if and only if T is a star.*

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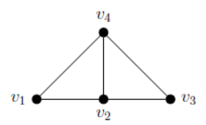
Throughout the following G denotes a connected graph with at least two vertices.

II. THE FORCING RESTRAINED STEINER NUMBER OF A GRAPH

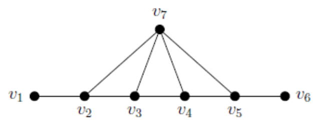
Even though every connected graph contains a minimum restrained Steiner set, some connected graph may contain several minimum restrained Steiner sets. For each minimum restrained Steiner set W in a connected graph G, there is always some subset T of W that uniquely determines W as the minimum restrained Steiner set containing T. We consider such "forcing subsets" in this section.

Definition 2.1 Let G be a connected graph and W a restrained Steiner set of G. Asubset $T \subseteq W$ is called a forcing subset for W if W is the unique minimum restrainedSteiner set containing T. A forcing subset for $W$of minimum cardinality is a minimumforcing subset of W. The forcing restrained Steiner number of W, denoted by $f_{rs}(W)$, isthe cardinality of a minimum forcing subset of W. The forcing restrained Steiner numberof G, denoted by $f_{rs}(G)$is $f_{rs}(G) = \min\{f_{rs}(W)\}$, where the minimum is taken over allminimum restrained Steiner sets W in G.

Example 2.2. For the graph G given in Figure 2.1, $W = \{v_1, v_3\}$ is the unique minimumrestrained Steiner set of G so that $f_{rs}(G) = 0$. For the graph G given in Figure 2.2,$W_1 = \{v_1, v_3, v_6\}$ and $W_2 = \{v_1, v_4, v_6\}$are the only two s_r- sets of G. It is clear that$f_{rs}(W_1) = 1$ and $f_{rs}(W_2) = 1$so that $f_{rs}(G) = 1$.



G
Figure 2.1



G
Figure 2.2

The next Theorem follows immediately from the definition of the restrained Steinernumber and the forcing restrained Steiner number of a connected graph G.

Theorem 2.3.*For every connected graph G, $0 \le f_{rs}(G) \le s_r(G)$.*

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the graph G given in Figure2.1, $f_{rs}(G) = 0$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 2.2, $f_{rs}(G) = 1$and $s_r(G) = 3$ so that $0 < f_{rs}(G) < s_r(G)$.

Theorem 2.5.*Let G be a connected graph. Then*

a) *$f_{rs}(G) = 0$if and only if G has a unique minimum restrained Steiner set.*
b) *$f_{rs}(G) = 1$ if and only if G has at least two minimum restrained Steiner sets, one*
of which is a unique minimum restrained Steiner set containing one of its elements,

and

c) *$f_{rs}(G) = s_r(G)$ if and only if no minimum restrained Steiner set of G is the unique minimum restrained Steiner set containing any of its proper subsets.*

Definition 2.6. A vertex v of a connected graph G is a *restrained Steiner vertex* of Gif v belongs to every restrained Steiner set of G. If G has a unique restrained Steiner set$W$, then every vertex of W is a restrained Steiner vertex of G.

Example 2.7. For the graph G given in Figure 2.2, $W_1 = \{v_1, v_3, v_6\}$ and $W_2 = \{v_1, v_4, v_6\}$ are the only two s_r - sets of G so that every s_r - set contains the vertices$v_1$and v_6. Hence the vertices v_1 and v_6 are the restrained Steiner vertices of G.

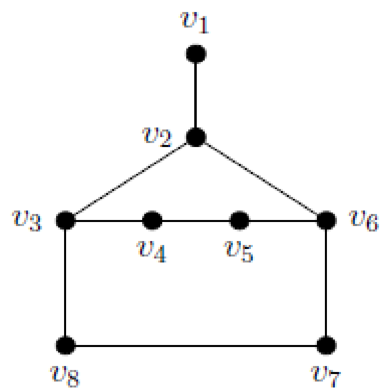
Theorem 2.8.*Let G be a connected graph and S be the set of all restrained Steinervertices of G. Then $f_{rs}(G) \le s_r(G) - |S|$.*

Proof. Let W be a minimum restrained Steiner set of G. Then $s_r(G) = |W|$. It is clearthat $S \subseteq W$. Let T be the unique minimum forcing restrained Steiner set containing$W - S$. Thus $f_{rs}(G) \le |W - S| \le |W| - |S| = s_r(G) - |S|$.

Corollary 2.9.*If G is a connected graph with k extreme vertices, then $f_{rs}(G) \le s_r(G) - k$.*

Proof. The proof follows from Theorems 1.1 and 2.8.

Remark 2.10. The bound in Theorem 2.8 is sharp. For the graph G given in Figure2.2, $W_1 = \{v_1, v_3, v_6\}$ and $W_2 = \{v_1, v_4, v_6\}$ are the only two s_r - sets of Gsuch that$f_{rs}(W_1) = 1$ and $f_{rs}(W_2) = 1$so that $f_{rs}(G) = 1$and $s_r(G) = 3$. Also, $S = \{v_1, v_6\}$ is theset of all restrained Steiner vertices of G and so $f_{rs}(G) = s_r(G) - |S|$. Also, the inequalityin Theorem 2.8 can be strict. For the graph G given in Figure 2.3, $W_1 = \{v_1, v_4, v_7\}$and$W_2 = \{v_1, v_5, v_8\}$ are the only two s_r - sets of G so that $s_r(G) = 3$. It is clear that$f_{rs}(W_1) = f_{rs}(W_2) = 1$so that $f_{rs}(G) = 1$. Now, v_1 is the only restrained Steiner vertexof G and so $|S| = 1$.Thus $f_{rs}(G) < s_r(G) - |S|$.



G
Figure 2.3

In the following we determine the forcing restrained Steiner number of some standardgraphs.

Theorem 2.11.*For the complete graph $K_p(p \geq 3, frsKp=0$.*

Proof. Since the set of all vertices of a complete graph K_p is the unique minimum restrained Steiner set, it follows from Theorem 2.5(a) that $f_{rs}(K_p) = 0$.

Theorem 2.12.*For the complete bipartite graph $K_{m,n}(m,n \geq 2), f_{rs}(K_{m,n}) = 0$.*

Proof. For the complete bipartite graph $K_{m,n}(m,n \geq 2)$, we have $s_r(K_{m,n}) = m + n$. Itfollows from Theorem 2.5(a) that $f_{rs}(K_{m,n}) = 0$.

Theorem 2.13.*Let G be a non trivial tree. Then $f_{rs}(G) = 0$.*

*Proof.*If G is not a star, then by Theorem 1.2, the set of all end vertices of G is theunique minimum restrained Steiner set of G. Now, it follows from Theorem 2.5(a) that$f_{rs}(G) = 0$.
If G is a star $K_{1,p-1}$, then it follows from Theorem 1.3, that the set of all vertices of$G$ is the unique minimum restrained Steiner set of G. By Theorem 2.5(a) it follows that$f_{rs}(G) = 0$.

Theorem 2.14.*For the cycle$C_p(p \geq 3)$, $f_{rs}(C_p) = \begin{cases} 0 \text{ for } p \in \{3,4,5\} \\ 1 \text{ for } p \geq 6 \text{ and } p \text{ is even.} \\ 2 \text{ for} p \geq 7 \text{ and} p \text{is odd} \end{cases}$*

Proof. It is proved in [8] that the restrained Steiner number $s_r(C_p) = p$ for $p \in \{3, 4, 5\}$ and so it follows from Theorem 2.5(a) that $f_{rs}(C_p) = 0$for $p \in \{3, 4, 5\}$. Let $p \geq 6$. Foran even cycle C_p, the restrained Steiner set consists of a pair of antipodal vertices of $C_p$and from Theorem 2.5(b) it follows that $f_{rs}(C_p) = 1$. Let p be odd with $p = 2n + 1, n = 1,2,3,\ldots$.
Let the cycle be $C_p: v_1, v_2, \ldots\ldots, v_n, v_{n+1}, v_{n+2}, \ldots\ldots, v_{2n+1}, v_1$. If $W = \{u, v\}$ isany set of two vertices of C_p, then no vertex of the $u - v$ longest path lies on the Steiner$W$ - tree in C_p and so no two element subset of C_p is a Steiner set of C_p. Now it is clearthat the sets $W_1 = \{v_1, v_{n+1}, v_{n+2}\}, W_2 = \{v_2, v_{n+2}, v_{n+3}\}, \ldots\ldots, W_{2n+1} = \{v_{2n+1}, v_n, v_{n+1}\}$ are the s- sets of C_p. Since $p \geq 7$, the subgraph$G[V - W_i](1 \leq i \leq 2n+1)$ has no isolatedvertices. Therefore the s- sets of C_p are the s_r - sets of C_p. It is clear from the s_r– setsW_i $(1 \leq i \leq 2n+1)$ that each $\{v_i\}(1 \leq i \leq 2n+1)$ is a subset of more than one s_r– setW_i. Hence it follows from Theorem 2.5(a) and (b) that $f_{rs}(C_p) \geq 2$. Now since v_{n+1} and$v_{n+2}$are antipodal to v_1, it is clear that W_1 is the unique s_r- set containing $\{v_{n+1}, v_{n+2}\}$and so $f_{rs}(C_p) = 2$.Hence the proof.

Theorem 2.15.*For the wheel $W_p = K_1 + C_{p-1}, (p \geq 5, srWp=p-3$ and $frsWp=p-4$.*

Proof.. Let $G = W_p$. Let v be the vertex of K_1 and let $v_1, v_2, \ldots\ldots, v_{p-1}, v_1$ be the cycle C_{p-1}. For $p = 5$,let $W_1 = \{v_1, v_3\}$and $W_2 = \{v_2, v_4\}$. Then $W_i, i = 1,2$ is a Steiner set of G. Since the subgraph$G[V - W_i]$, $i = 1,2]$ has no isolated vertices, $W_i, i = 1,2$ is a restrained Steiner set of G so that $s_r(W_p) = 2 = p - 3$and $f_{rs}(W_p) = 1 = p - 4$. Let $p \geq 6$. Let W be a Steiner set of G. If $v \in W$, then$\langle W \rangle$ is connected. Then the Steiner W - tree contains elements of W only. Therefore$v \notin W$.Hence $W \subseteq V(C_{p-1})$. Let W be any subset of vertices of C_{p-1} of cardinality $p -$ 3obtained by deleting two non-adjacent vertices of C_{p-1}. We may assume without lossof generality that $W = \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_{p-1}\}$, where$1 \leq i < j \leq p-1$ and $j \geq i+2$. Since the subgraph induced by $G[V - W]$ has noisolated vertices, it is clear that W is a minimun restrained Steiner set of G so that$s_r(W_p) \leq |W| = p - 3$. Let W'be any restrained Steiner set with$|W'| \leq p - 4$ Thenat least three vertices of C_{p-1} do not belong to W' . If these vertices are consecutiveon C_{p-1}, then W' is not a Steiner set of G. Otherwise there are non adjacent verticesv_k and $v_l(k < l)$ such that $v_k, v_l \notin W'$. Since the size of any Steiner W' - tree is atmost $p - 4$, it is easily seen that v_k and $v_l$do not lie on any Steiner W' - tree of G andso W' is not a restrained Steiner set of G. Thus $s_r(W_p) = p - 3$. Since the subgraphinduced by a proper Steiner set is disconnected, it follows that any s_r- set is of the
form $W = \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_{p-1}\}$, where v_i and v_j arenon-adjacent. Let T be a subset of W with $|T| \leq p - 5$. Since $p \geq 6$, there exists distinctvertices $x, y \in W$ such that $x, y \notin T$. If x and y are adjacent, then x is non-adjacent to atleast one of $v_i$and v_j , sayv_j . Then $W_1 = V(C_{p-1}) - \{x, v_j\}$ is a s_r - set such that $W_1 \neq W$and W_1 properly contains T. If x and y are non-adjacent, then $W_2 = V(C_{p-1}) - \{x, y\}$ isa s_r- set such that $W_2 \neq W$ and W_2 properly contains T. Thus T is not a forcing subsetfor W. Now, we show that there exists a forcing subset of W of cardinality $p - 4$. Forconvenience, let$W = \{v_2, v_4, v_5, v_6, \ldots, v_{p-1}\}$. We show that $T_1 = \{v_4, v_5, v_6, \ldots, v_{p-1}\}$is a forcing subset of W. If T_1 is not a forcing subset for W, then there exists as_r-set $W' \neq W$such that $T_1 \subseteq W'$. Since $W' \neq W, |W'| = p - 3$ and $|T_1| = p - 4, W'$must contain exactly one of $v_1$or v_3. In any case, $\langle W' \rangle$is connected and so W' is not arestrained Steiner set of G, which is a contradiction. Hence $f_{rs}(W_p) = p - 4$.

Theorem 2.16.*If $W = \{u, v\}$ is a s_r - set of a connected graph G, then u and v aretwo antipodal vertices of G.*

Proof. Let $W = \{u, v\}$ be a s_r - set of G. Then every vertex of G lies on a Steiner W- tree of G. Since every Steiner W - tree is a $u - v$ geodesic, every vertex of G lies ona geodesic joining u and v. Also the subgraph$G[V - W]$ has no isolated vertices. Weclaim that $d(u,v) = d(G)$. If$d(u,v) < d(G)$, then let x and y be two vertices of G suchthat $d(x,y) = d(G)$. Now, it follows that x and y lie on distinct geodesics joining $u$and$v$. Hence

$d(u,v) = d(u,x) + d(x,v)$(1)
and
$d(u,v) = d(u,y) + d(y,v)$
(2)
By the triangle inequality,
$d(x,y) \leq d(x,u) + d(u,y)$(3)

Since $d(u,v) < d(x,y)$, (3) becomes
$d(u,v) < d(x,u) + d(u,y)$
(4)

Using (4) in (1), we get $d(x,v) < d(x,u) + d(u,y) - d(u,x) = d(u,y)$. Thus,
$d(x,v) < d(u,y)$ (5)
Also, by triangle inequality, we have
$d(x,y) \leq d(x,v) + d(v,y)$
(6)
Now, using (5) and (2), (6) becomes $d(x,y) < d(u,y) + d(v,y) = d(u,v)$. Thus, $d(G) < d(u,v)$which is a contradiction.Hence $d(u,v) = d(G)$ so that u and v are antipodalvertices of G.

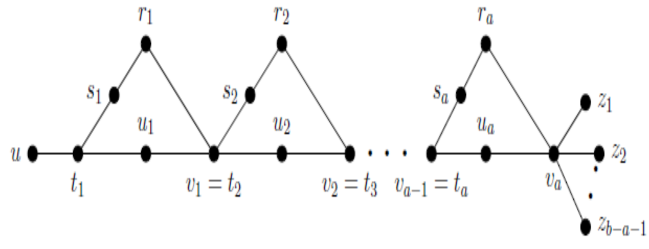
Theorem 2.17.*If G is a connected graph with $s_r(G) = 2$, then $f_{rs}(G) \leq 1$.*

Proof. Let $W = \{u,v\}$ be any s_r - set of G. Then by Theorem 2.16, u and v are antipodalvertices of G. Suppose that $f_{rs}(G) = 2$. Then it follows from Theorem 2.5 (c) that W isnot the unique s_r - set containing u and so there exists $x \neq u$ *in*G such that $W' = \{u,x\}$is also a s_r - set of G. By Theorem 2.16, u and x are two antipodal vertices of G and v is an internal vertex of some $u - x$ geodesic in G. Therefore, $d(u,v) < d(u,x)$, which isa contradiction. Hence $f_{rs}(G) \leq 1$.

In view of Theorem 2.3, the following theorem gives a realization for the forcingrestrained Steiner number and the restrained Steiner number of a graph.

Theorem 2.18.*For every pair a, b of integers with $0 \leq a < b, b \geq 2$, there exists aconnected graph G such that $f_{rs}(G) = a$and $s_r(G) = b$.*

*Proof.*Suppose $a = 0$. Let $G = K_{1,b-1}$. Then by Theorem 2.13, $f_{rs}(G) = 0$ and fromTheorem 1.3, $s_r(G) = b$. Now, assume that $a \geq 1$. For $b = a + 1$, let $G = K_1 + C_{a+3}, (a \geq 1)$. By Theorem 2.15, $s_r(G) = a + 1 = b$and $f_{rs}(G) = a$. For $b \neq a + 1$, let$F_i : r_i, s_i, t_i, u_i, v_i, r_i (1 \leq i \leq a)$ be a copy of the cycle C_5. Let G be the graph obtainedfrom $F_i(1 \leq i \leq a)$by first identifying the vertices v_{i-1} of F_{i-1} and $t_i$of$F_i(2 \leq i \leq a)$and then adding $b - a$new vertices $u, z_1, z_2, \dots, z_{b-a-1}$ and joining the $b - a$ edges ut_1 and$v_a z_i (1 \leq i \leq b - a - 1)$. The graph G is given in Figure 2.4. Let$Z = \{u, z_1, z_2, \dots, z_{b-a-1}\}$
be the set of all end-vertices of G. Let $H_i = \{r_i, s_i\} (1 \leq i \leq a)$. First we show that$s_r(G) = b$. By Theorem 1.1, Z is a subset of every restrained Steiner set of G and so$s_r(G) \geq b - a$. Since $S(Z) \neq V$, Z is not a restrained Steiner set of G. We observethat every s_r- set of G must contain exactly one vertex from each$H_i (1 \leq i \leq a)$ and so$s_r(G) \geq b - a + a = b$. On the other hand, since the set $W = Z \cup \{r_1, r_2, \dots, r_a\}$ is aminimum restrained Steiner set of G, it follows that $s_r(G) \leq |W| = b$. Thus $s_r(G) = b$.



G
Figure 2.4

Next we show that $f_{rs}(G) = a$. Since every s_r- set of Gcontains Z, it follows fromTheorem 2.8 that $f_{rs}(G) \leq s_r(G) - |Z| = b - (b - a) = a$. Now, since $s_r(G) = b$and everyminimum restrained Steiner set of G contains Z, it is easily seen that every minimumrestrained Steiner set S is of the form$Z \cup \{c_1, c_2, \dots, c_a\}$, where$c_i \in H_i, (1 \leq i \leq a)$. Let$T$ be any proper subset of $S$with $|T| < a$. Then there is a vertex $c_j (1 \leq i \leq a)$ such that$c_j \notin T$. Let d_j be a vertex of H_j distinct from c_j. Then $S_1 = (S - \{c_j\}) \cup \{d_j\}$ is as_r-setproperly containing T. Thus S is not the unique s_r- set containing T and so $T$is not aforcing subset of S. This is true for all s_r- sets of G and so it follows that $f_{rs}(G) = a$

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