International Journal of Engineering and Advanced Technology (IJEAT)
ISSN: 2249-8958 (Online), Volume-8 Issue-6S3, September 2019

Generalized Gaussian Jacobsthal and Generalized Gaussian Jacobsthal Lucas Polynomial

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***Abstract*: The aim of the present paper is to present a generalization of Gaussian Jacobsthal polynomial and Gaussian Jacobsthal Lucas polynomial. Present paper extends the work of Asci and Gurel [3]. Some important generalizations of Generating function, Binet formula, Explicit formula, Q matrix and determinantal representations of these polynomials are also produced.**

Keywords: Generalized Gaussian Jacobsthal; Generalized Gaussian Jacobsthal Lucas Polynomial; Binet formula; Cassini Identity

I. INTRODUCTION

In 1977, Berzsenyi [4] initiated the theory of Gaussian Fibonacci numbers. Afterwards different authors formulated various interesting results using Gaussian Fibonacci and Gaussian Lucas polynomials. On the other hand, Asci and Gurel [1] introduced Bivariate Gaussian Fibonacci and Bivariate Gaussian Lucas polynomials. In the same work, authors [Asci and Gurel] gave generating functions, Binet formulas, explicit formulas and Q matrix for these polynomials. Simillarly, Asci and Gurel [2] defined Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers and using these notions they discussed some interesting results. Recently, Asci and Gurel [3] defined the Gaussian Jacobsthal polynomial and Gaussian Jacobsthal Lucas polynomials and formulated generating function, Binet formula, Explicit formulas and determinantal representations for these polynomials.

Horadam [7] introduced the Jacobsthal and the Jacobsthal Lucas sequences under the following recurrence relations:

$$J_n = J_{n-1} + 2J_{n-2},\ n \geq 2$$

where $J_0 = 0$ and $J_1 = 1$, and

$$j_n = j_{n-1} + 2j_{n-2},\ n \geq 2$$

where $j_0 = 2$ and $j_1 = 1$.

In the same work Horadam [7], Jacobsthal and the Jacobsthal Lucas polynomial sequences were defined by using the following recurrence relations:

$$J_n(x) = J_{n-1}(x) + 2J_{n-2}(x),\ n \geq 2$$

where $J_0 = 0$ and $J_1 = 1$, and

$$j_n(x) = j_{n-1}(x) + 2j_{n-2}(x),\ n \geq 2$$

where $j_0 = 2x$ and $j_1 = 1$.

Subsequently, the Gaussian Jacobsthal sequence and Gaussian Jacobsthal Lucas sequence were defined by Asci and Gurel [2] as follows:

$$GJ_{n+1} = GJ_n + 2GJ_{n-1},\ n \geq 1$$

where $GJ_0 = \frac{i}{2}$ and $GJ_1 = 1$, and

$$Gj_{n+1} = Gj_n + 2Gj_{n-1},\ n \geq 1$$

where $Gj_0 = 2 - \frac{i}{2}$ and $Gj_1 = 1+2i$.

The Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomial in [3] were defined by

$$GJ_{n+1}(x) = GJ_n(x) + 2GJ_{n-1}(x),\ n \geq 1$$

where $GJ_0(x) = \frac{i}{2}$ and $GJ_1(x) = 1$, and

$$Gj_{n+1}(x) = Gj_n(x) + 2Gj_{n-1}(x),\ n \geq 1$$

where $Gj_0(x) = 2 - \frac{i}{2}$ and $Gj_1(x) = 1+2i$.

Further, in [3] the authors have also formulated the following notions:

Generating functions:

(i) $g(t,x) = \sum_{n=0}^{\infty} GJ_n(x).t^n = \frac{2t+i(1-t)}{2-2t-4t^2x}$;

(ii) $h(t,x) = \sum_{n=0}^{\infty} Gj_n(x).t^n = \frac{4-2t+i(t-1+4xt)}{2-2t-4t^2x}$.

Binet Formula:

For $n \geq 0$,

(i) $GJ_n(x) = \frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)} + i.x\frac{\alpha^{n-1}(x)-\beta^{n-1}(x)}{\alpha(x)-\beta(x)}$;

(ii) $Gj_n(x) = \alpha^n(x) + \beta^n(x) + i.x.(\alpha^{n-1}(x) + \beta^{n-1}(x))$.

Explicit Formula:

(i) $GJ_n(x) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-r-1}{r} (2.x)^r + i\sum_{r=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-r-2}{r} (2)^r.(x)^{r+1}$;

(ii) $Gj_n(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-r}\binom{n-r}{r} (2.x)^r + i\sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-r-1}\binom{n-r-1}{r} (2)^r.(x)^{r+1}$.

Cassini Identity:

For $n \geq 1$,

$GJ_{n-1}(x).GJ_{n+1}(x) - GJ_n^2(x) = (-1)^n(2)^{n-2}.x^{n-1}(2+x-i)$.

In this paper, we generalized the work of Asci and Gurel in [13] by introducing the Generalized Gaussian Jacobsthal polynomial, Generalized Gaussian Lucas polynomial and thereby we obtain generating function and Cassini identity and introduce Q matrix for these polynomials. We also present explicit formula and determinantal representation, and give various identities using these polynomials.

Manuscript published on 30 September 2019.
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II. GENERALIZED GAUSSIAN JACOBSTHAL AND GENERALIZED GAUSSIAN JACOBSTHAL LUCAS POLYNOMIALS.

In this section we give our notions and formulate our main results.

Definition 2.1. The Generalized Gaussian Jacobsthal polynomial $\{GJ_{k,n}(x)\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$GJ_{k,n+1}(x) = GJ_{k,n}(x) + 2^k.x\, GJ_{k,n-1}(x)$, $n \geq 1$ and k is non – negative integer

with initial conditions

$GJ_{k,0}(x) = \frac{i}{2}$ and $GJ_{k,1}(x) = 1$.

Definition 2.2. The Generalized Gaussian Jacobsthal Lucas polynomial $\{Gj_{k,n}(x)\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$Gj_{k,n+1}(x) = Gj_{k,n}(x) + 2^k.x\, Gj_{k,n-1}(x)$, $n \geq 1$, and k is non – negative integer

with initial conditions

$Gj_{k,0}(x) = 2 - \frac{i}{2}$ and $Gj_{k,1}(x) = 1+2xi$.

It is interesting to note that

$GJ_{k,n}(1) = GJ_{k,n}$ and $Gj_{k,n}(1) = Gj_{k,n}$

where $GJ_{k,n}$ and $Gj_{k,n}$ are the Generalized Gaussian Jacobsthal and Generalized Gaussian Jacobsthal Lucas number.

Next, we give few number of terms of these sequences in the following tables:

Generalized Gaussian Jacobsthal Polynomials:

N	$GJ_{k,n}$
0	$\frac{i}{2}$
1	1
2	$1+ i2^{k-1}.x$
3	$(1+ 2^k.x) + i2^{k-1}.x$
4	$(1+ 2^{k+1}x) + i(2^{k-1} + 2^{2k-1}).x$
5	$1+ (2^k+2^{k+1})x +2^{2k}.x^2 + i(2^{k-1} + 2^{2k-1}.x + 2^{2k-1}).x$
6	$1+ 2^{k+2}.x + (2^{2k}+2^{2k+1}).x^2 + ix(2^{k-1} + 2^{2k-1}.x + 2^{2k}.x + 2^{3k-1}).x$
⋮	⋮

Generalized Gaussian Jacobsthal Lucas Polynomials:

N	$Gj_{k,n}$
0	$2 - \frac{i}{2}$
1	$1+2xi$
2	$(1+2^{k+1}.x)+ ix(2 - 2^{k-1})$
3	$(1+ 2^{k+1} + 2^k x) + ix(2 - 2^{k-1} + 2^{k+1}.x)$
4	$(1+ 2^{k+2}x + 2^{2k-1}x^2) + ix(2 - 2^{k-1} + 2^{k+2}x - 2^{2k-1}.x)$
5	$(1+ 2^{k+2}x + 2^{2k+2}.x^2 + 2^{2k}.x^2)+ix(2 - 2^{k-1} + 2^{k+2}x - 2^{2k}.x + 2^{k+1}x - 2^{2k+1}.x^2)$
6	$(1+2^{k+2}x+2^{2k+3}.x^2 + 2^k.x + 2^{3k+1}.x^3)+$ $ix(2 - 2^{k-1} + 2^{k+2}x + 2^{2k+1}.x^2 - 2^{2k-1}x + 2^{2k+3}.x^2 - 2^{3k-1}.x^2)$
⋮	⋮

The main recursive properties of the generalized Gaussian Jacobsthal and Generalized Gaussian Lucas polynomial are determined in this section. Initially, we can begin with the characteristic equation, that is

$$t^2 - t - 2^k x = 0$$

$$t = \frac{1 \pm \sqrt{1+4.2^k.x}}{2}$$

$$t = \frac{1 \pm \sqrt{1+2^{k+2}.x}}{2}$$

$\alpha(x) = \frac{1+\sqrt{1+2^{k+2}.x}}{2}$ and $\beta(x) = \frac{1-\sqrt{1+2^{k+2}.x}}{2}$

$\alpha(x) + \beta(x) = 1$, $\alpha(x).\beta(x) = -2^k.x$, $\alpha(x) - \beta(x) = \sqrt{1+2^{k+2}.x}$

Some Properties:

Theorem 2.1. (i) The Generating function for Generalized Gaussian Jacobsthal polynomial is

$g(t,x) = \sum_{n=0}^{\infty} GJ_{k,n}(x).t^n = \frac{2t+ i(1-t)}{2-2t-2^{k+1}t^2x}$

and

(ii) For Generalised Gaussian Jacobthal Lucas polynomial is

$h(t,x) = \sum_{n=0}^{\infty} Gj_{k,n}(x).t^n = \frac{4-2t+ i(t-1+4xt)}{2-2t-2^{k+1}t^2x}$

Proof: (i) Let g(t,x) be the generating function of generalized Gaussian Jacobsthal polynomial sequence $GJ_{k,n}(x)$ then

$g(t,x) - t.g(t,x) - 2^k.x.\, t^2\, g(t,x) = \sum_{n=0}^{\infty} GJ_{k,n}(x).t^n - \sum_{n=0}^{\infty} GJ_{k,n}(x).t^{n+1} - 2^k.x \sum_{n=0}^{\infty} GJ_{kn}(x).t^{n+2}$

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$$= GJ_{k,0}(x) + GJ_{k,1}(x).t - GJ_{k,0}(x)t + \sum_{n=2}^{\infty} t^n [GJ_{k,n}(x) - GJ_{k,n-1}(x) - 2^k GJ_{k,n-2}(x)]$$
$$= \frac{i}{2} + (1-\frac{i}{2})t$$
$$= \frac{2i + i(1-t)}{2}$$

By taking g(t,x) paranthesis, we get

$$(1-t-2^k.x.t^2)\, g(t,x) = \frac{2i + i(1-t)}{2}$$
$$g(t,x) = \frac{2i + i(1-t)}{2(1-t-2^k.x.t^2)}$$
$$g(t,x) = \frac{2i + i(1-t)}{(2-2t-2^{k+1}.x.t^2)}$$

(ii) Let h(t,x) be the generalized function of generalized Gaussian Jacobsthal Lucas polynomial sequence $Gj_{k,n}(x)$ then

$$h(t,x) - t.h(t,x) - 2^k.x.t^2\, h(t,x) = \sum_{n=0}^{\infty} Gj_{k,n}(x).t^n - \sum_{n=0}^{\infty} Gj_{k,n}(x).t^{n+1} - 2^k.x \sum_{n=0}^{\infty} Gj_{k,n}(x).t^{n+2}$$
$$= Gj_{k,0}(x) + Gj_{k,1}(x).t - Gj_{k,0}(x)t + \sum_{n=2}^{\infty} t^n [Gj_{k,n}(x) - Gj_{k,n-1}(x) - 2^k Gj_{k,n-2}(x)]$$
$$= 2-\frac{i}{2} + (1+2xi) - (2-\frac{i}{2})t$$
$$= 2-\frac{i}{2} + 1+2xi - 2t + \frac{i}{2}t,$$

that is,

$$(1-t-2^k.x.t^2)\, h(t,x) = \frac{4 - i - 2t - 4xti - 4t + it}{2}$$

which implies,

$$h(t,x) = \frac{4 - 2t + i(4xt - 1 + t)}{2(1-t-2^k.x.t^2)}$$

and hence $h(t,x) = \frac{4 - 2t + i(4xt - 1 + t)}{2 - 2t - 2^{k+1}.x.t^2}$

The following results hold immediately by applying induction:

Theorem 2.2. (Generalized Binet Formula)

For $n \geq 0$

(i) $GJ_{k,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + i.x.2^{k-1} \frac{\alpha^{n-1}(x) - \beta^{n-1}(x)}{\alpha(x) - \beta(x)}$

(ii) $(i)\ Gj_{k,n}(x) = \alpha^n(x) + \beta^n(x) + i.x.2^{k-1}(\alpha^{n-1}(x) + \beta^{n-1}(x))$

Theorem 2.3. The explicit formula of Generalized Gaussian Jacobsthal polynomial is

$$GJ_{k,n}(x) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-r-1}{r} (2^k.x)^r + i \sum_{r=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-r-2}{r} (2^k)^r.(x)^{r+1}$$

Theorem 2.4. The explicit formula of Generalized Gaussian Jacobsthal Lucas polynomial is

$$Gj_{k,n}(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-r} \binom{n-r}{r} (2^k.x)^r + i \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-r-1} \binom{n-r-1}{r} (2^k)^r.(x)^{r+1}$$

Theorem 2.5. Let $D_n(x)$ denote the $n \times n$ tridiagonal matrix as

$$D_n(x) = \begin{bmatrix} 1 & i2^{k-1} & 0 & \cdots & 0 \\ -x & 1 & 2^k x & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 2^k x \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \ n \geq 1$$

and let $D_0(x) = \frac{i}{2}$, Then $\det D_n(x) = GJ_{k,n}(x)$, $n \geq 1$

Proof: We shall prove the result by using mathematical induction on n.

For n = 1 and n = 2

$$\det D_1(x) = 1 = GJ_{k,1}(x)$$

and $\det D_2(x) = 1 + i2^{k-1}x = GJ_{k,2}(x)$

Assume that the theorem is true for n−1 and n−2, that is

$$\det D_{n-1}(x) = GJ_{k,n-1}(x)$$

and $\det D_{n-2}(x) = GJ_{k,n-2}(x)$

Now, we shall prove that the theorem for n

For,

$$\det D_n(x) = \det D_{n-1}(x) + 2^k.x \det D_{n-2}(x)$$
$$= GJ_{k,n-1}(x) + 2^k.x\, GJ_{k,n-2}(x)$$
$$= GJ_{k,n}(x)$$

Theorem 2.6. Let $H_n(x)$ denote the $n \times n$ tridiagonal matrix as

$$H_n(x) = \begin{bmatrix} 2-\frac{i}{2} & \frac{x}{2}-1 & 0 & \cdots & 0 \\ 1 & ix & 2^k x & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 2^k x \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \ n \geq 1$$

Then, $\det H_n(x) = GJ_{k,n-1}(x)$, $n \geq 0$.

Proof: We shall prove the result by using mathematical induction on n.

For n = 1 and n = 2

$$\det H_1(x) = 2 - \frac{i}{2} = Gj_{k,0}(x)$$

and $\det H_2(x) = 1 + 2x\, i = Gj_{k,1}(x)$

Assume that the theorem is true for n−1 and n−2

$$\det H_{n-1}(x) = Gj_{k,n-2}(x)$$

and $\det H_{n-2}(x) = Gj_{k,n-3}(x)$

Now, we shall prove that the theorem for n

For,

$$\det H_n(x) = \det H_{n-1}(x) + 2^k.x \det H_{n-2}(x)$$
$$= Gj_{k,n-1}(x) + 2^k.x\, Gj_{k,n-2}(x)$$
$$= Gj_{k,n}(x)$$

Now, we introduce the matrices Q(x), P and R that plays the role of the Q-Matrix of Fibonacci numbers. Let Q(x), P and R denotes the 2×2 matrices defined as

$$Q(x) = \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 + ix2^{k-1} & 1 \\ 1 & \frac{i}{2} \end{bmatrix}, \quad R = \begin{bmatrix} 1 + 2^{k+1}.x + ix(2-2^{k-1}) & 1 + 2ix \\ 1 + 2ix & 2 - \frac{i}{2} \end{bmatrix}.$$

Theorem 2.7. Let $n \geq 1$, Then

$$Q^n(x).P = \begin{bmatrix} GJ_{k,n+2}(x) & GJ_{k,n+1}(x) \\ GJ_{k,n+1}(x) & GJ_{k,n}(x) \end{bmatrix}$$

where $GJ_{k,n}(x)$ is the nth Generalized Gaussian Jacobsthal Polynomial.

Proof:- We shall prove the result by using mathematical induction on n.

For n = 1

$$Q(x).P = \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 + ix2^{k-1} & 1 \\ 1 & \frac{i}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 1 + i2^{k-1}.x + 2^k x & 1 + i2^{k-1}x \\ 1 + i2^{k-1}x & 1 \end{bmatrix}$$
$$= \begin{bmatrix} GJ_{k,3}(x) & GJ_{k,2}(x) \\ GJ_{k,2}(x) & GJ_{k,1}(x) \end{bmatrix}$$

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Assume that the theorem hold for n = t , that is

$$[Q(x)]^t.P = \begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}^t \begin{bmatrix}1+ix2^{k-1} & 1\\ 1 & \frac{i}{2}\end{bmatrix} = \begin{bmatrix}GJ_{k,t+2}(x) & GJ_{k,t+1}(x)\\ GJ_{k,t+1}(x) & GJ_{k,t}(x)\end{bmatrix}$$

Now, we shall prove the theorem for n = t+1, we have

$$[Q(x)]^{t+1}.P = \begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}^{t+1} \begin{bmatrix}1+ix2^{k-1} & 1\\ 1 & \frac{i}{2}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}^t \begin{bmatrix}1+ix2^{k-1} & 1\\ 1 & \frac{i}{2}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}.\begin{bmatrix}GJ_{k,t+2}(x) & GJ_{k,t+1}(x)\\ GJ_{k,t+1}(x) & GJ_{k,t}(x)\end{bmatrix}$$

$$= \begin{bmatrix}GJ_{k,t+3}(x) & GJ_{k,t+2}(x)\\ GJ_{k,t+2}(x) & GJ_{k,t+1}(x)\end{bmatrix}$$

Theorem 2.8. Let $n \geq 1$, Then

$$Q^n(x).R = \begin{bmatrix}Gj_{k,n+2}(x) & Gj_{k,n+1}(x)\\ Gj_{k,n+1}(x) & Gj_{k,n}(x)\end{bmatrix}$$

where $Gj_{k,n}(x)$ is the nth Generalized Gaussian Jacobsthal Lucas Polynomial.

Proof: We shall prove the result by using mathematical induction on n.

For n = 1

$$Q(x).R = \begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}.\begin{bmatrix}1+2^{k+1}.x+ix(2-2^{k-1}) & 1+2ix\\ 1+2ix & 2-\frac{i}{2}\end{bmatrix}$$

$$= \begin{bmatrix}1+2^{k+1}.x+2^k.x+ix(2-2^{k-1}+2^{k+1}x) & 1+2ix+2^{k+1}x-i.2^{k-1}.x\\ 1+2ix+2^{k+1}x-i.2^{k-1}.x & 1+2xi\end{bmatrix}$$

$$= \begin{bmatrix}1+2^{k+1}.x+2^k.x+ix(2-2^{k-1}+2^{k+1}x) & 1+2^{k+1}x+i.x(2-2^{k-1})\\ 1+2^{k+1}x+i.x(2-2^{k-1}) & 1+2xi\end{bmatrix}$$

$$= \begin{bmatrix}Gj_{k,3}(x) & Gj_{k,2}(x)\\ Gj_{k,2}(x) & Gj_{k,1}(x)\end{bmatrix}$$

Assume that the theorem holds for n = t, that is

$$[Q(x)]^t .R = \begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}^t.\begin{bmatrix}1+2^{k+1}.x+ix(2-2^{k-1}) & 1+2ix\\ 1+2ix & 2-\frac{i}{2}\end{bmatrix} = \begin{bmatrix}Gj_{k,t+2}(x) & Gj_{k,t+1}(x)\\ Gj_{k,t+1}(x) & Gj_{k,t}(x)\end{bmatrix}$$

Now, we shall prove that the theorem for n = t+1,

For,

$$[Q(x)]^{t+1}.R = \begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}^{t+1}.\begin{bmatrix}1+2^{k+1}.x+ix(2-2^{k-1}) & 1+2ix\\ 1+2ix & 2-\frac{i}{2}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}.\begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}^t . \begin{bmatrix}1+2^{k+1}.x+ix(2-2^{k-1}) & 1+2ix\\ 1+2ix & 2-\frac{i}{2}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2^k x\\ 1 & 0\end{bmatrix}.\begin{bmatrix}Gj_{k,t+2}(x) & Gj_{k,t+1}(x)\\ Gj_{k,t+1}(x) & Gj_{k,t}(x)\end{bmatrix}$$

$$= \begin{bmatrix}Gj_{k,t+3}(x) & Gj_{k,t+2}(x)\\ Gj_{k,t+2}(x) & Gj_{k,t+1}(x)\end{bmatrix}$$

Theorem 2.9. (Generalized Cassini Identity)

For $n \geq 1$

$$GJ_{k,n-1}(x).GJ_{k,n+1}(x) - GJ_{k,n}^2(x) = (-1)^n(2^k)^{n-2}.x^{n-1}(2^{k-1}x+2-i)$$

Proof: We can prove the theorem by matrix method. For, first of all we shall determine the determinants of matrices.

$$\det Q^{n-1}(x) = \begin{vmatrix}1 & 2^k.x\\ 1 & 0\end{vmatrix}^{n-1} = (-2^k x)^{n-1}.$$

$$\det P = \begin{vmatrix}1+i.2^{k-1}.x & 1\\ 1 & \frac{i}{2}\end{vmatrix} = \frac{i}{2} - \frac{2^{k-1}}{2}x - 2 = -\frac{1}{2}[2^{k-1}.x + 2 - i]$$

By Theorem 2.7,

$$[Q(x)]^{n-1}.p = \begin{bmatrix}GJ_{k,n+1}(x) & GJ_{k,n}(x)\\ GJ_{k,n}(x) & GJ_{k,n-1}(x)\end{bmatrix}.$$

We get the determinants of the matrices

$$GJ_{k,n-1}(x).GJ_{k,n+1}(x) - GJ_{k,n}^2(x) = \det(Q^{n-1}(x).P)$$
$$= \det(Q^{n-1}(x)).\det(P)$$
$$= (-2)^k.x)^{n-1}.\left(-\frac{1}{2}[2^{k-1}.x + 2 - i]\right)$$
$$= (-1)^n.(2^k)^{n-2}.x^{n-1}(2^{k-1}.x+2-i)$$

Theorem 2.10. For $n \geq 1$

$$Gj_{k,n-1}(x).Gj_{k,n+1}(x) - Gj_{k,n}^2(x) = (-1)^{n-1}.2^{n-2}.x^{n-1}[(8x)^2 + (2+3.2^{k-1})x + 2 - i2^{k+2}.x]$$

Proof: We can prove the theorem by matrix method. For, first of all we determine the determinants of matrices.

$$\det Q^{n-1}(x) = \begin{vmatrix}1 & 2^k.x\\ 1 & 0\end{vmatrix}^{n-1} = (-2^k x)^{n-1}$$

$$\det R = \begin{vmatrix}1+2^{k+1}.x+ix(2-2^{k-1}) & 1+2ix\\ 1+2ix & 2-\frac{i}{2}\end{vmatrix}$$
$$= [(8x)^2 + (2+3.2^{k-1})x + 2 - i - i2^{k+2}.x]$$

By Theorem 2.8,

$$Q^{n-1}(x).R = \begin{bmatrix}Gj_{k,n+1}(x) & Gj_{k,n}(x)\\ Gj_{k,n}(x) & Gj_{k,n-1}(x)\end{bmatrix}$$

We get the determinants of the matrices

$$Gj_{k,n-1}(x).Gj_{k,n+1}(x) - Gj_{k,n}^2(x) = \det(Q^{n-1}(x).R)$$
$$= \det(Q^{n-1}(x)).\det(R)$$

=

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$$(-2)^k.x)^{n-1}.\frac{1}{2}[(8x)^2 + (2+3.2^{k-1})x + 2 - i - i2^{k+2}.x]$$

$$= (-1)^{n-1}.2^{n-2}.x^{n-1}[(8x)^2 + (2+3.2^{k-1})x + 2 - i - i2^{k+2}.x]$$

Theorem 2.11. The sums of the Generalized Gaussian Jacobsthal Polynomial and Generalized Gaussian Jacobsthal Lucas Polynomial are given as:

(i) $\sum_{m=0}^{n} GJ_{k,m}(x) = \frac{1}{2^k.x}[GJ_{k,n+2}(x) - 1]$;

(ii) $\sum_{m=0}^{n} Gj_{k,m}(x) = \frac{1}{2^k.x}[Gj_{k,n+2}(x) - (1+2xi)]$.

Proof: (i) For $n \geq 1$, we have

$$GJ_{k,n+1}(x) = GJ_{k,n}(x) + 2^k.x\, GJ_{k,n-1}(x),$$
$$GJ_{k,n-1}(x) = \frac{1}{2^k.x}[GJ_{k,n+1}(x) - GJ_{k,n}(x)].$$

For the last equation

$$GJ_{k,0}(x) = \frac{1}{2^k.x}[GJ_{k,2}(x) - GJ_{k,1}(x)],$$
$$GJ_{k,1}(x) = \frac{1}{2^k.x}[GJ_{k,3}(x) - GJ_{k,2}(x)],$$
$$GJ_{k,2}(x) = \frac{1}{2^k.x}[GJ_{k,4}(x) - GJ_{k,3}(x)],$$
$$\vdots \qquad\qquad \vdots$$
$$GJ_{k,n-1}(x) = \frac{1}{2^k.x}[GJ_{k,n+1}(x) - GJ_{k,n}(x)],$$
$$GJ_{k,n}(x) = \frac{1}{2^k.x}[GJ_{k,n+2}(x) - GJ_{k,n+1}(x)].$$

On adding all of the above equation, can get

$$\sum_{m=0}^{n} GJ_{k,m}(x) = \frac{1}{2^k.x}[GJ_{k,n+2}(x) - GJ_{k,1}(x)]$$
$$= \frac{1}{2^k.x}[GJ_{k,n+2}(x) - 1].$$

(ii) For $n \geq 1$, we have

$$Gj_{k,n+1}(x) = Gj_{k,n}(x) + 2^k.x\, Gj_{k,n-1}(x);$$
$$Gj_{k,n-1}(x) = \frac{1}{2^k.x}[Gj_{k,n+1}(x) - Gj_{k,n}(x)].$$

For the last equation

$$Gj_{k,0}(x) = \frac{1}{2^k.x}[Gj_{k,2}(x) - Gj_{k,1}(x)],$$
$$Gj_{k,1}(x) = \frac{1}{2^k.x}[Gj_{k,3}(x) - Gj_{k,2}(x)],$$
$$Gj_{k,2}(x) = \frac{1}{2^k.x}[Gj_{k,4}(x) - Gj_{k,3}(x)],$$
$$\vdots \qquad\qquad \vdots$$
$$Gj_{k,n-1}(x) = \frac{1}{2^k.x}[Gj_{k,n+1}(x) - Gj_{k,n}(x)],$$
$$Gj_{k,n}(x) = \frac{1}{2^k.x}[Gj_{k,n+2}(x) - Gj_{k,n+1}(x)].$$

On adding all of the above equation, can get

$$\sum_{m=0}^{n} Gj_{k,m}(x) = \frac{1}{2^k.x}[Gj_{k,n+2}(x) - Gj_{k,1}(x)],$$
$$= \frac{1}{2^k.x}[Gj_{k,n+2}(x) - (1+2xi)].$$

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