Dominations In Semigraphs

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Abstract: The semigraph generalization is more closely related to the axiom that the two edges in a semigraph have at most one vertex in common, where as the hypergraph generalization is based on the consideration of an edge as a subset of two elements of the set of vertices of graphs. The concept of domination is an important parameter in graph theory. They provide a lot of space to the theoretical development of graphs and their applications. The dominations in semigraphs have no exception. In this paper, we study various dominations of semigraphs arising out of the corresponding adjacencies exist in semigraphs.

Index Terms: $a -$ Domination, $ca -$ Domination, $e -$ Domination, $le -$ Domination, Independent Domination, Semigraph.

I. INTRODUCTION

The dominating set plays an important role in computer and communication networks. Mainly mobile network, Wireless Sensor network and social network.

Let $S$ is a pair $(V, X)$ with nonempty set of vertices $S$ and set of ordered n-tuples $n \geq 2$ of different vertices called edges of $S$ satisfying the following conditions hold good:

i. every two edges have at most one vertex in common place.

ii. the edges $E_1 = (u_1, u_2, \ldots, u_n)$ and $E_2 = (v_1, v_2, \ldots, v_n)$ are said to be identical iff

a. $m = n$ and

b. either $u_i = v_i$ or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

Consider a semigraph $S$ where $V = \{v_1, v_2, \ldots, v_9\}$ and $X = \{(v_1, v_2, v_3), (v_4, v_5, v_6), (v_6, v_7, v_8), (v_1, v_8)\}

The vertices $v_1, v_3, v_6, v_9$ are end vertices, $v_4, v_5$ are middle vertices and $v_2, v_7, v_9$ are middle-end vertices.

The order and size of a semigraph $S$ is respectively denoted by the numbers $|V|$ and $|X|$. The number of vertices in $E$ is represented as $|E|$, we also write, $V = V_e \cup V_m \cup V_{me}$ where $V_e$, $V_m$ and $V_{me}$ respectively represent the set of end, middle and middle-end vertices.


II. PROPOSED METHODOLOGY

In this paper, $a -$ domination, $ca -$ domination, $e -$ domination, $le -$ domination of path semigraphs and cycle semigraphs. Based on that, we find the above four types of dominations for semigraphs as well as independent dominations for all the semigraphs.

III. TYPES OF DOMINATIONS IN SEMIGRAPH

A. Four Types of Vertex Dominations in Semigraphs

B. Definition

A subset $D$ of $V$ is called $a -$ dominating set of $S$ if for all $v \in V - D$ there exist $u \in D$ such that $u$ and $v$ are adjacent. The minimum cardinality of an $a -$ dominating set is called $a -$ domination number of $S$ and is represented by $\gamma_a(S)$. If $D$ is $a -$ dominating set of $S$ and it has no proper subset is called minimal $a -$ dominating set.

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C. a-Dominations in Semigraphs

A vertex $u$ in $D$ is said to be $a$-isolate in $D$, if no $v$ in $D$ such that $u$ and $v$ are $a$-adjacent.

D. Lemma

An $a$-dominating set $D \subseteq V_m$ is minimal iff every $u \in D$ is an $a$-isolate vertex in $D$.

The following is an important theorem in domination theory.

E. Ores Theorem[12]

An $a$-dominating set $D \subseteq V$ of $S$ is minimal iff for all $u \in D$ one of the properties following holds good:

i. $u$ is an $a$-isolate vertex of $D$.

ii. there exists another $V$ in $V$ such that $N_a(v) \cap D = \{u\}$.

F. Lemma

$\gamma_a(S-e) = \gamma_a(S) + k$, where $0 \leq k \leq |e| - 1$.

Proof:

Consider a semigraph $S$, $v$ is a minimal $a$-dominating set and $|V| = n$. Suppose if $S$ is disconnected and has a component contains only an edge $e$. In this case, consider the semigraph $S-e$. Surely, $v$ contains exactly one vertex say $u$ of $e$. In $S-e$, all vertices of $e$ become $a$-isolates and hence all vertices of $e$ are strongly dominated vertices in $S-e$. Suppose $e$ contains the vertices $U = \{u, u_1, u_2, ..., u_m\}$ then $V \cup U$ is a minimal dominating set. Hence $\gamma_a(S-e) = |V| + |U| - 1$, since $u$ occurs twice as a member of $V$ and also as a member of $U$.

Hence $\gamma_a(S-e) = |V| + |U| - 1 = \gamma_a(S) + |e| - 1$.

Therefore, $k = |e| - 1$, where the edge $e$ is a component in the disconnected semigraph $S$.

Suppose, if $u$ is a strongly dominated vertex in $e$ then in this case also $\gamma_a(S-e) = \gamma_a(S) + |e| - 1$. Otherwise, if $u$ is a strongly $a$-dominated vertex but not in edge $e$ then $\gamma_a(S-e) = \gamma_a(S)$, i.e., $k = 0$. This is also true when $u$ is a weakly $a$-dominated. If $u$ is a weakly $a$-dominated, and $v$ is the alternative vertex $u$, and $v$ is $a$-adjacent to the all the vertices of $e$, then $\gamma_a(S-e) = \gamma_a(S)$. Suppose if there are $m$ vertices in $e$, not adjacent to $v$ then $\gamma_a(S-e) = \gamma_a(S) + m$, where $m < |e|$ - number of vertices in $e$ adjacent to $v$.

G. Lemma

Let $S = (V, X)$ be a simple connected semigraph only with $p$ number of principal edges then $\gamma_a(S) \leq \left\lceil \frac{p}{2} \right\rceil$.

Equality arises iff $S$ is a cycle semigraph.

Proof:

Consider a simple connected semigraph containing only $p$ number of principal edges. Then we have two cases.

i. there are exactly two end vertices of degree one.

ii. the degree of all the end vertices are of degree two.

Suppose if the semigraphs do not have any middle-end vertices, then they are simply paths or cycles. In this case $\gamma_a(S) < \left\lceil \frac{p}{2} \right\rceil$. If every edge has at least one middle vertex then $\gamma_a(S) = \left\lceil \frac{p}{2} \right\rceil$.

H. Theorem

$\gamma_a(S) = \gamma(S_a)$

Proof:

Let $S$ be a semigraph. Also let $\gamma_a(S) = n$, and $D = \{v_1, v_2, ..., v_n\}$ is a minimal $a$-dominating set of $S$. Either $v_i, i = 1, 2, ..., n$ is an $a$-isolate in $D$ otherwise there exists another $v_j$ in $D$ such that $N_a(v_j) \cap D = \{v_i\}$. In other words, every $v_i$ in $D$ is either $a$-isolate or there is at most only one vertex in $D$ in the neighbourhood of $v_i$. Let $S_a$ be the adjacency graph of $S$. Any two vertices are adjacent in $S_a$ iff the vertices are adjacent in $S$. If the vertices $v_1, v_2, ..., v_n$ are all end or middle-end vertices in $S$, then they are also in $S_a$, and form a minimal dominating set in $S_a$. Suppose, if a vertex $v_i$ in $S$ is a middle vertex in $E_i = (v_i, v_{i+1}, ..., v_{i,...,v_i})$, then we show that replacing $v_i$ with $v_s$ or $v_i$ in $D$ does not affect the minimum cardinality of the set $D$. If $v_i$ is $a$-isolate in $D$, then the including $v_s$ or $v_i$ in place of $v_i$ in $D$, play the role of dominating set with minimum cardinality. Note that if $v_i$ is not an $a$-isolate in $D$, and there is a $v_j$ in $D$ such that $N(v_j) \cap D = \{v_i\}$, then $v_j$ definitely should be either a middle vertex or a middle-end vertex, otherwise, $D-\{v_j\}$ is also a $a$-dominating set. Hence, $v_j$ must be an end vertex or middle-end vertex. In this case, replace $v_i$ with the other end vertex present in $E_i$. Hence $\gamma_a(S) = \gamma(S_a)$.

I. Observations

If $S$ has no $a$-isolates then $\gamma_a(S) \leq \frac{n}{2}$, where

$n = \left\lceil \frac{|V(S)|}{2} \right\rceil$. In particular

i. $\gamma_a(S) \leq \left\lceil \frac{|V(S)|}{2} \right\rceil$.

ii. If
\[ \deg(v) = |V_v(S)| - 1, \ v \in V_v(S), \] then \( \gamma_v(S) = 1. \)

iii. In particular, \( \gamma_a(S) = \frac{|V_v(S)|}{2} \), if \( |V_v(S)| \) is even, and \( \deg(v) = 2 \), for all \( v \in V_v(S) \).

**J. Lemma**

\[
\gamma_a(P_{s(n)}) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
\frac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof:**

Consider a path semigraph \( P_{s(n)} \) with \( n = 2k \), \( k = 1,2, \ldots \) edges, say \( E_1, E_2, \ldots, E_{2k} \), and \( 2k + 1 \) end vertices say \( v_1, v_2, \ldots, v_{2k+1} \), such that \( v_1, v_{i+1} \) are the end vertices of the edge \( E_i \). First, it can be noticed that no \( a \)-dominating set containing middle vertices cannot be a minimal \( a \)-dominating set. Let \( u_i \in E_i \) is a middle vertex then the set \( D = \{u_1, u_2, \ldots, u_{2k}\} \) is a \( a \)-dominating set in \( P_{s(n)} \). Since, each neighbourhood of middle vertex contains all other vertices in that particular edge, the set \( D \) is a weakly \( a \)-dominating set in \( P_{s(n)} \).

Also the vertex in \( u_i \), \( a \)-dominates only the vertices present in the edge \( E_i \), and none of the vertices in the adjacent edges \( E_{i-1} \) or \( E_{i+1} \). It can also be seen that both of the \( u_i \), \( u_{i+1} \in N(V_{i+1}) \), where \( i = 2r - 1, \ r = 1,2, \ldots, k \). Hence \( D' = \{v_2, v_4, \ldots, v_{2k}\} \) is a minimal \( a \)-dominating set and therefore \( \gamma(P_{s(n)}) = k = \frac{n}{2} \).

Suppose, if \( n = 2k - 1, \ k = 1,2, \ldots \) then \( P_{s(n)} \) will have \( 2k - 1 \) edges, say \( E_1, E_2, \ldots, E_{2k-1} \) and \( 2k \) end vertices \( v_1, v_2, \ldots, v_{2k} \) each such that \( v_i, v_{i+1} \) are the end vertices. By similar argument as above it can be found that \( D' = \{v_2, v_4, \ldots, v_{2k}\} \) is a minimal \( a \)-dominating set. Hence \( \gamma(P_{s(n)}) = k = \frac{n+1}{2} \).

**K. Lemma**

\[
\gamma_a(C_{s(n)}) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
\frac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof:**

Let \( n = 2k \), then \( P_{s(n)} \) has \( n + 1 \) end vertices and \( C_{s(n)} \) has \( n \) end vertices. The only difference between \( P_{s(n)} \) and \( C_{s(n)} \) is that the edges \( E_1 \) and \( E_n \) are adjacent in \( C_{s(n)} \) but \( E_1 \) and \( E_n \) has one end vertex with degree one. The semigraphs \( P_{s(n)} \) has \( 2k + 1 \) end vertices and \( C_{s(n)} \) has \( 2k \) end vertices. In particular the vertex \( V_{2k+1} \) is present in \( P_{s(n)} \), but it is not in \( C_{s(n)} \). Hence the strong \( a \)-dominating set \( D' = \{v_2, v_4, \ldots, v_{2k}\} \) of \( P_{s(n)} \) is also \( a \)-dominating set in \( C_{s(n)} \).

Similarly we can see that the weak dominating set \( D' = \{v_1, v_3, \ldots, v_{2k-1}\} \) of \( P_{s(n)} \), when \( n \) is odd which is also a minimal \( a \)-dominating set of \( C_{s(n)} \).

**L. Definition**

An \( a \)-independent dominating set is a set of vertices in \( S \) which is both \( a \)-dominating and \( a \)-independent. The minimum size of an \( a \)-independent dominating set is called \( a \)-independent domination number which is noted as \( i_a(S) \). Similarly we can define \( i_e(S) \), \( i_{ce}(S) \) and \( i_{le}(S) \). R.B. Allan et. Al [2] studied independent domination number in 1978.

**M. Lemma**

\[
i_a(P_{s(n)}) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
\frac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof:**

The minimal \( a \)-dominating set identified for \( P_{s(n)} \), is also a minimal \( a \)-independent set. Hence, for path semigraph the independent number is same as \( a \)-domination number. Hence the lemma.

**N. Lemma**

\[
i_a(C_{s(n)}) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
\frac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof:**

We know that cycle semigraph \( C_{s(n)} \) can be obtained from the path semigraph \( P_{s(n)} \), by making \( E_i \) and \( E_n \) are adjacent.

The minimal \( a \)-dominating set identified for \( P_{s(n)} \) when \( n \) is even, is also a \( a \)-independent set. Therefore, \( a \)-domination number is same as independent number for \( C_{s(n)} \), when \( n \) is even. But the minimal \( a \)-dominating set obtained for \( P_{s(n)} \), when \( n \) is odd is also a minimal \( a \)-dominating set for \( C_{s(n)} \), the set is also independent in \( P_{s(n)} \), but it is not independent set in \( C_{s(n)} \).
In $P_{s(n)}$, when $n$ is odd, the minimal $a -$ dominating set is the weak $a -$ dominating set $D' = \{v_1, v_3, \ldots, v_{2k-1}\}$ and the end vertices $v_1$ and $v_{2k-1}$ are adjacent in $C_{s(n)}$. It can be seen that the vertex $v_{2k-1}$ is included in $D'$, only to cover the vertices present in $E_{2k-2}$. Hence if we include only middle vertex in $E_{2k-2}$, in place of $v_{2k-1}$ in $D'$, the resulting set is also a $a -$ dominating set, and also an independent set. Hence the $a -$ domination number is same as independent number in $C_{s(n)}$, when $n$ is odd. Hence the lemma.

IV. CA-DOMINATIONS IN SEMIGRAPHS

Let $S$ be a semigraph. A subset $D$ of $V$ is called a $ca -$ dominating set, if for all vertex $v \in V - D$, there exists $u \in D$ such that $u$ and $v$ are adjacent and consecutive as well. The set $V$ is also called as $\gamma_{ca} -$ set of $G$.

The minimum number of a $ca -$ dominating set is called as $ca -$ domination number of $S$, which is represented by $\gamma_{ca}(S)$.

A. Lemma

For any semigraph $S$ , $\gamma_{ca}(S) = \gamma(S_{ca})$.

Proof : 

We know that $u \ , \ v$ are consecutively adjacent if the vertices are adjacent (i.e., they belong to same edge) and consecutive as well. Also we know that $u \ , \ v$ are adjacent in $S_{ca}$ if the vertices are consecutively adjacent in $S$. Hence $|V(S)| = |V(S_{ca})|$.

Also, for $u \in S$, if $N_{ca}(u) = \{u_1, \ldots, u_s\}$ in $S$, then $N(u)$ is also the set $\{u_1, \ldots, u_s\}$ in $S_{ca}$, and vice versa. Therefore, if $D = \{v_1, v_2, \ldots, v_n\}$ is a minimal $ca -$ dominating set in $S$ , $D = \{v_1, v_2, \ldots, v_n\}$ is a dominating set in $S_{ca}$. Hence $\gamma_{ca}(S) = \gamma(S_{ca})$.

B. Lemma

For any semigraph $S$ , $\gamma_{a}(S) \leq \gamma_{ca}(S)$.

C. Lemma

If $m$ is the maximum number of middle vertices in an edge, then for $k = 1,2,3,$...

$$\gamma_{ca}(P_{s(n)}) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } m = 3k - 1 \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } m = 3k \quad \text{where } k = 1,2, \ldots \\ n\left\lceil \frac{m}{3} \right\rceil & \text{if } m = 3k + 1 \end{cases}$$

D. Lemma

$$\gamma_{ca}(C_{s(n)}) = \begin{cases} \left\lceil \frac{m}{3} \right\rceil + 2 & \text{if } n = 3k \\ n\left\lceil \frac{m}{3} \right\rceil & \text{if } n = 3k - 1,3k - 2 \end{cases}$$

where $k = 1,2,3,$...

V. E AND IE- DOMINATIONS IN SEMIGRAPHS

A. Definition

Consider a semigraph. A subset $D$ of $V$ is called an $e -$ dominating set, if for an end vertex $v \in V - D$, there exists $u \in D$ such that $u$ and $v$ are end vertices of an edge. $\gamma_e(S)$ is the $e -$ domination number of $S$.

B. Lemma

$$\gamma_e(P_{s(1)}) = \gamma_e(P_{s(2)}) = 1$$

$$\gamma_e(P_{s(n)}) = \begin{cases} \frac{n}{3} + 1 & \text{if } n = 3k, \\ \frac{n}{3} & \text{if } n = 3k + 1,3k + 2 \end{cases}$$

C. Lemma

$$\gamma_e(C_{s(n)}) = \left\lceil \frac{n}{3} \right\rceil$$

D. Definition

Let $S$ be a semigraph. A subset $D$ of $V$ is called an $le -$ dominating set, if for an end vertex $v \in V - D$, there exists $u \in D$ such that $u$ and $v$ belong to the same edge and at least one of them is an end vertex of that edge. $\gamma_{le}(S)$ is the minimum $le -$ domination number of $S$.

E. Lemma

$$\gamma_{le}(P_{s(n)}) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n}{2} + 1 & \text{if } n \text{ is odd} \end{cases}$$

$$\gamma_{le}(C_{s(n)}) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n}{2} + 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof : 

The proof is similar to $a -$ dominating set. Since, in $a -$ dominating set alternate end vertex dominates all the other vertices. Similarly, in $le -$ dominating set alternate end vertices is enough for dominate all the end vertices. Hence the following lemma is discussed in [12].

F. Lemma

$$\gamma_{a}(S) = \gamma_{le}(S)$$
G. Lemma

Consider a semigraph $S = (V, X)$ such that $|X| = 1$; $|V| = n$. Then $\gamma_a(S) = \gamma_e(S) = \gamma_{le}(S) = 1$ and $\gamma_{ca}(S) = \lceil \frac{n}{3} \rceil$.

Proof:

Let $X = |E|$, and $E = \{u_1, u_2, \ldots, u_n\}$. Then $D = \{u_1\}$ is a minimal dominating set with respect to the definitions of adjacent, le - adjacent, and e - adjacent.

It can be found that the sets $D_1 = \{u_2, u_3, \ldots, u_n\}$, $D_2 = \{u_2, u_4, \ldots, u_{n-1}\}$, $D_3 = \{u_2, u_5, \ldots, u_{n-2}\}$ are minimal ca - dominating sets when $n = 3r + 1, 3r + 2, 3r + 3$, $r = 2, 3, 4$, respectively.

Also when $n = 2, 3$ the set $\{u_2\}$ and $n = 4, 5$ the set $\{u_2, u_3\}$ and $n = 6$, the set $\{u_2, u_4\}$ are minimal ca - dominating sets. Hence $\gamma_{ca}(S) = \lceil \frac{n}{3} \rceil$.

VI. RESULTS

In this paper, the generalized dominating sets for path and cycle semigraphs are calculated and proved and it shown below.

\[
\gamma_a(P_{(n)}) = \gamma_e(C_{(n)}) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}
\]

\[
i_a(P_{(n)}) = i_e(C_{(n)}) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}
\]

\[
\gamma_{ca}(P_{(n)}) = \begin{cases} \frac{n+1}{3} & \text{if } m = 3k - 1 \\ \frac{n}{3} & \text{if } m = 3k \\ \frac{n}{3} & \text{if } m = 3k + 1 \end{cases}
\]

where $k = 1, 2, \ldots$

\[
\gamma_{ca}(C_{(n)}) = \begin{cases} \frac{n+1}{3} & \text{if } n = 3k \\ \frac{n}{3} & \text{if } n = 3k - 1, 3k - 2 \end{cases}
\]

\[
\gamma_{e}(P_{(n)}) = \gamma_{e}(C_{(n)}) = 1
\]

\[
\gamma_{e}(C_{(n)}) = \begin{cases} \frac{n}{3} & \text{if } n = 3k \\ \frac{n}{3} & \text{if } n = 3k + 1, 3k + 2 \end{cases}
\]

where $k = 1, 2, \ldots$

\[
\gamma_{le}(P_{(n)}) = \gamma_{le}(C_{(n)}) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}
\]

VII. CONCLUSION

The four types of domination numbers are defined in this paper. Special properties of these domination numbers and independent domination numbers have been studied. The nature of domination numbers of some special semigraphs have also been discussed.
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