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Wavelet Solution for Nonlinear Reaction-Diffusion Equations

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Abstract: ***In this paper, we consider Fisher's equation to find the approximate solution to overcome the difficulty to handle its nonlinearity. For solving this nonlinear PDE, we propose a method based on Legendre wavelets with lesser number of connection coefficients. We also study the theoretical analysis and error bound for the proposed technique. Two examples are tested with the proposed method to show the applicability and efficiency. The outcomes show that this approach fulfils the error bound conditions.***

.Index Terms*: *Fisher's equation; Legendre wavelets; Convergence Analysis; Reaction-diffusion.

I. INTRODUCTION

The nonlinear differential equations play a pivotal role in the areas of science and engineering, and some of the nonlinear partial differential equations are non-integrable ones. One of the nonlinear equations considered here namely Fisher's equation[1,2]takes the different form

$$u_t = u_{xx} + \alpha\left(1-u^{\beta}\right)(u-a) \quad (1)$$

with $u(x,0) = f(x)$.

When $\alpha = u$ and $\beta = 1$ in Eq. (1), it becomes

$$u_t = u_{xx} + u(1-u)(u-a),\ 0 < a < 1 \quad (2)$$

When $a = 0\, and\, \beta = 1$ in Eq. (1), we get the Fisher's equation

$$u_t = u_{xx} + \alpha\, u(1-u). \quad (3)$$

and the generalized Fisher's equation, encountered in population growth model is

$$u_t = u_{xx} + \alpha\, u\left(1-u^{\beta}\right) \quad (4)$$

Generally solving a nonlinear equation is a tedious task. To solve these kinds of equations, we have approximate, numerical and semi-analytical techniques [2-4]. At the same time, each method has its own merits and demerits based on their applicability. Not all nonlinear problems have been solved by a single approach. These methods have deviated on their nature of the nonlinearity, initial and boundary conditions. Symmetries and transformations methods are also used to find the exact solutions for the same without considering their initial conditions. So the researchers keep on trying the new methods to overcome the limitations stated in the nonlinear differential equations.

Many researchers have investigated the Fisher's equation through various methods to solve, and it can be found in [2-8].Fisher's equation is one of the nonlinear equation encountered in various physical and chemical phenomena and very helpful in modeling certain parameters like reaction-diffusion equations. Chemical kinetics, Population dynamics, neurophysiology are the areas where this equation plays a pivotal role. In the general form, for different values or expressions of the parameters α, β and 'a', we get the variants of Fisher's equation encountered in different studies. In this paper, we solve the Fisher's equation through Legendre wavelet-based hybrid method. Wavelet-based methods are usually applied in image processing, restoration, compression etc. The orthogonal properties of the various wavelets are effectively used to solve the differential equations in nonlinear nature.

The subsequent sections of the article are structured as follows: In section 2, we describe the basics of Legendre polynomials, wavelets and function approximation. Section 3 is devoted to the proposed approach for solving Eq. (1). In Section 4, the convergence and error analysis of the same is given. Numerical examples and concluding remarks are presented in section 5 and 6.

II. LEGENDRE WAVELETS AND FUNCTION APPROXIMATION

The Legendre polynomialsP_m (t) of order m on the interval [-1,1] is defined by:

$$P_0(t) = 1 \ , \qquad P_1(t) = t$$

$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t\, P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t),$$

$$m = 1,2,3,....$$

From the Legendre polynomials, we define the Legendre wavelets [9] of the form $\psi_{nm}(t) = \psi\left(k, \hat{n}, m, t\right)$ have four arguments:

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$\hat{n}=2n-1,\ n=1,2,3\ldots,2^{k-1}$,

k can assume any positive integer. They are defined on the interval [0,1) as

$$\psi_{nm}(t)=\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}P_m\left(2^k t-\hat{n}\right), & \text{for } \frac{\hat{n}-1}{2^k}\le t\le\frac{\hat{n}+1}{2^k}, \\ 0 & , \text{otherwise}\end{cases} \tag{5}$$

where m = 0,1,2, … , M-1, n = 1,2,3, … , 2^{k-1}.

Any function u(x,t) with infinite terms is defined over [0,1)x [0,1) is expanded as

$$u(x,t)=C^T\ \Psi\ \Phi_n \tag{6}$$

where $\Phi_n = t^{n-1}$ and

$$c_{nm}=(2m-1)\int_0^1\int_0^1 u(x,t)\psi_{nm}(x)\phi_n(t)\,dx\,dt.$$

Hence

$$u_{nm}(x,t)\cong\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x)\ \phi_n(t)=C^T\ \Psi(x)\ \Phi_n(t)$$

(7)

where C and $\psi(t)$ are 2^{k-1}M x 1matrices of unknown and known.

Proposed scheme for Fisher's equation

Consider Eq. (1)

$$\int_0^t u_t\,dt=\int_0^t u_{xx}dt+\alpha\int_0^t\left(1-u^\beta\right)(u-a)dt.$$

$$u(x,t)-u(x,0)=\int_0^t u_{xx}dt+\alpha\int_0^t\left(1-u^\beta\right)(u-a)dt$$

.

$$u(x,t)=u(x,0)+\int_0^t u_{xx}dt+\alpha\int_0^t\left(1-u^\beta\right)(u-a)dt$$

. (8)

By using (7), we have

$$\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x)\,\phi_n(t)=u(x,0)+\int_0^t\left(C^T\psi(x)\,t^{n-1}\right)_{xx}dt$$

$$+\alpha\int_0^t\left(1-\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x)\,\phi_n(t)\right)^\beta\right)\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x)\,\phi_n(t)-a\right)dt. \tag{9}$$

Collocating Eq. (9) at 2^{k-1}M points, we get

$$\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x_i)\,\phi_n(t)=u(x,0)+\int_0^t\left(C^T\psi(x_i)\,t^{n-1}\right)_{xx}dt$$

$$+\alpha\int_0^t\left(1-\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x_i)\,\phi_n(t)\right)^\beta\right)\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x_i)\,\phi_n(t)-a\right)dt. \tag{10}$$

The value of C can be obtained by solving from2^{k-1}M nonlinear equations.

III. THEORETICAL ANALYSIS

In this section, we discuss the theoretical background for the function approximation defined in the previous section through the process of convergence and error estimation. We prove the idea of convergence by restricting the given connection coefficients by some constants, and thereby it converges to the original solution uniformly. For uniform convergence, the δ value depends only on ε but not on *x*. For error estimate, we go with the difference of the original approximation with the truncated series in the approximation. Hence the estimate or the bound is attained for those terms that remain after truncation. The bound value that is obtained is validated through some examples, and the error value sticks to the upper bound. This error bound expression falls to zero uniformly when the value of M gets increased.

Theorem 1 (Convergence Theorem)

The function $u_{nm}(x,t)$ converges towards the exact solution $u(x,t)$ where $u(x,t)$ is continuous and bounded in the second derivative.

Proof:

$$\left|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(x_i)\,\phi_n(t)\right|\le\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}|c_{nm}|\,|\psi_{nm}(x_i)|\,|\phi_n(t)|$$

$$\le\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}|c_{nm}|$$

We can compute the coefficients c_{nm} through $2^{k-1}M$ system of equations.

Theorem 2 (Error Analysis)

The error between exact and approximate solution is given by

$$\varepsilon_{mn}\le 2^{\frac{-(k+1)}{2}}\alpha\sqrt{6}\,2^k\sum_{i=2^k}^{\infty}\sum_{j=M}^{\infty}\frac{1}{\sqrt{2j+1}}\frac{1}{\sqrt{2^k(j-2)+2i-3}}\frac{1}{\sqrt{2i-1}}$$

where $\left|\frac{\partial^2 u}{\partial x^2}\right|\le\alpha$ on [0,1) x [0,1) and

$$\varepsilon_{mn}=\left\|\left[u(x,t)-u_{nm}(x,t)\right]\right\|_{L^2}.$$

IV. ILLUSTRATIVE EXAMPLES

In this section, we consider the following problems discussed in [5]

Example 1:

In this case, we will consider Eq. (1) for

$\alpha=6;\beta=1;a=0$

$$u_t=u_{xx}+6u(1-u),\quad 0<x<1 \tag{11}$$

subject to $u(x,0)=\dfrac{1}{(1+e^x)^2}$.

We set K=3 and M=4, the values of the coefficients are

$C_{10}=\frac{7}{48}$; $C_{11}=\frac{-3}{16\sqrt{3}}$; $C_{12}=\frac{-1}{48\sqrt{5}}$;$C_{13}=\frac{-1}{80\sqrt{7}}$;

$C_{20}=\frac{-2295}{384\sqrt{2}}$; $C_{21}=\frac{157}{128\sqrt{6}}$; $C_{22}=\frac{125}{384\sqrt{10}}$; $C_{23}=\frac{15}{640\sqrt{14}}$;

$C_{30}=\frac{-45975}{3072\sqrt{2}}$; $C_{31}=\frac{-17415}{3072\sqrt{6}}$; $C_{32}=\frac{-975}{3072\sqrt{10}}$; $C_{33}=\frac{-5}{1024\sqrt{14}}$;

$C_{40}=\frac{7393425}{3072\sqrt{2}}$; $C_{41}=\frac{1892465}{3072\sqrt{6}}$;$C_{42}=\frac{61275}{3072\sqrt{10}}$;$C_{43}=\frac{-555}{1024\sqrt{14}}$.

This infinite series coincides with the exact solution $u(x,t)=\frac{1}{\left(1+e^{x-5t}\right)^2}$ for higher values of M. The maximum error for this approach is

$$2^{\frac{-(k+1)}{2}}\alpha\sqrt{6}\,2^k\sum_{i=2^k}^{\infty}\sum_{j=M}^{\infty}\frac{1}{\sqrt{2j+1}}\frac{1}{\sqrt{2^k(j-2)+2i-3}}\frac{1}{\sqrt{2i-1}}$$

The convergence of the method is demonstrated in Figure 1 for M=10.For each value of t ranging from 0.1 to 0.9, the error between exact and LWM solution at x=0.25 and t from 0.1 to 0.9 has been displayed in Figure 2, and the maximum error satisfies the error bound.

Example 2:

In this example, we consider Eq. (1) for $\alpha=1;\beta=6;a=0$ the subject to $u(x,0)=\frac{1}{\left(1+e^{\left(\frac{3}{2}\right)x}\right)^{\frac{1}{3}}}$

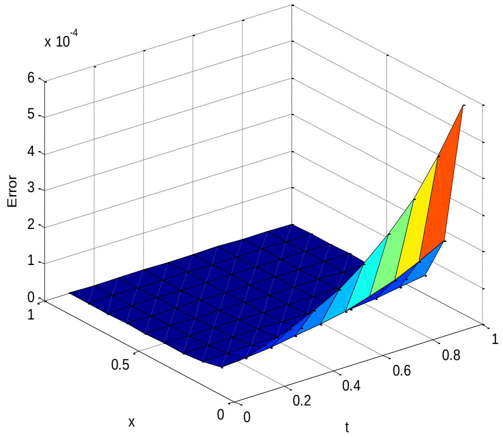

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Figure. 1.The plot of absolute error with M=10

By using LWM discussed in this paper, we get the series solution

$$u(x,t)=\frac{1}{\left(1+e^{\left(\frac{3}{2}\right)x}\right)^{\frac{1}{3}}}+\frac{5e^{\left(\frac{3}{2}\right)x}}{4\left(1+e^{\left(\frac{3}{2}\right)x}\right)^{\frac{1}{3}}}+\frac{25e^{\frac{3}{2}x}\left(e^{\frac{3}{2}x}-3\right)}{16\left(1+e^{\left(\frac{3}{2}\right)x}\right)^{\frac{7}{3}}}\frac{t^2}{2!}+\dots$$

and the exact solution is [10]

$$u(x,t)=\left\{\frac{1}{2}\tanh\left[-\frac{3}{4}\left(x-\frac{5}{2}t\right)\right]+\frac{1}{2}\right\}^{\frac{1}{3}}$$

Comparison report for example 2 has been elucidated in table 1. In a similar manner, we can show that the exact solution for the generalized Fisher's equation

$u_t=u_{xx}+u\left(1-u^{\beta}\right)$ is

$$u(x,t)=\left\{\frac{1}{2}\tanh\left[-\frac{\beta}{2\sqrt{2\beta+4}}\left(x-\frac{\beta+4}{\sqrt{2\beta+4}}t\right)+\frac{b}{2}\right]+\frac{1}{2}\right\}^{\frac{2}{\beta}},$$

which is the same as the solution found by Wang [11].

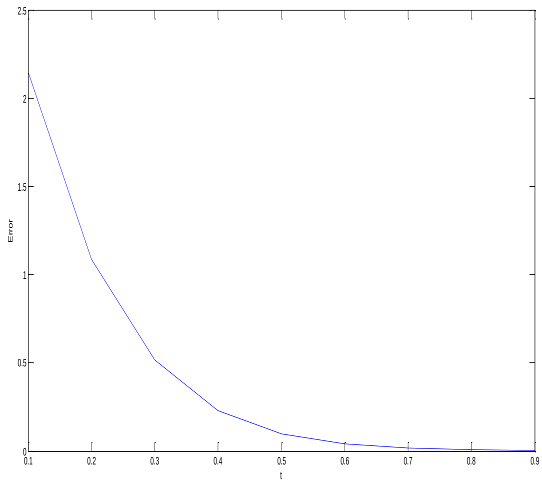

Figure. 2. The maximum absolute error for t.

Table 1: LWM solution vs exact solution

x	t	LWM	Exact
0.25	0.10	0.7936	0.7937
	0.15	0.8175	0.8177
	0.20	0.8402	0.8400
	0.25	0.8601	0.8604
0.50	0.10	0.7411	0.7413
	0.15	0.7685	0.7682
	0.20	0.7936	0.7937
	0.25	0.8176	0.8177
0.75	0.10	0.6847	0.6846
	0.15	0.7132	0.7133
	0.20	0.7415	0.7413
	0.25	0.7682	0.7682

Example 3:

Consider $u_t=u_{xx}+u(1-u),\ 0<x<1$

subject to a constant initial condition $u(x,0)=\lambda$.

For K=3 and M=4, we get the following connection coefficients values.

$C_{10}=\lambda$; $C_{11}=0$; $C_{12}=0$;$C_{13}=0$;

$C_{20}=\frac{\lambda-\lambda^2}{\sqrt{2}}$; $C_{21}=0$; $C_{22}=0$; $C_{23}=0$

$C_{30}=\frac{\lambda-3\lambda^2+2\lambda^3}{2\sqrt{2}}$; $C_{31}=0$; $C_{32}=0$; $C_{33}=0$;

$C_{40}=\frac{\lambda-7\lambda^2+12\lambda^3-6\lambda^4}{6\sqrt{2}}$; $C_{41}=0$;$C_{42}=0$;$C_{43}=0$.

The solution is

$$u(x,t)=\lambda+\frac{(\lambda-\lambda^2)}{\sqrt{2}}\sqrt{2}\ t+\frac{(\lambda-3\lambda^2+2\lambda^3)}{2\sqrt{2}}\sqrt{2}\ t^2+\frac{(\lambda-7\lambda^2+12\lambda^3-6\lambda^4)}{6\sqrt{2}}\sqrt{2}\ t^3.$$

For larger M, we get $u(x,t)=\dfrac{\lambda e^t}{1-\lambda+\lambda e^t}$.

V. CONCLUSION

Fisher's equation has been solved by the wavelet method. The method is based on the function approximation technique where one group of variables is based on the usual wavelets, and the other group is based on algebraic polynomials. Hence by this new approximation technique, the number of coefficients employed gets reduced, and still, the solution quality seems to be an appreciable one when comparing with other methods. Also, the theoretical analysis has been carried out. The efficiency of this method is tested with several examples, and it requires only a lesser number of connection coefficients. The idea of convergence has been proved by restricting the given connection coefficients by some constants, and thereby it converges to the original solution uniformly. For uniform convergence, the δ value depends only on ε but not on x. Hence the estimate or the bound has been calculated for those terms that remain after truncation. The bound value that is obtained is validated through some examples, and the error value sticks to the upper bound. This error bound expression falls to zero uniformly when the value of M gets increased.

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