

Calculation of Latitude by Secant Method on the Basis of Spatial Orthogonal Coordinates

Pavel Aleksandrovich Medvedev, Anatoly Ivanovich Uvarov

Abstract: This article discusses the importance of conversion of orthogonal coordinates X, Y, Z into curvilinear geodesic coordinates: latitude B , longitude L , and altitude H . Geodesic latitude in tangent function is calculated by transcend equation with variable coefficients which is meaningless when the point is located on rotation axis of Earth ellipsoid. For this case the algorithm for calculation of latitude, altitude, and longitude is provided. The algorithms are developed and analyzed by solution of less complicated transcend equation with the reduced latitude u using root isolating. Reasonability of development of non-iterative algorithms is substantiated. Root isolating for $tg u$ made it possible to apply the secant method in the segment $[T_1, T_2]$ according to which the latitude is calculated with the error of $\Delta u \leq 0,0017'' (T_1 = tg u_1; T_2 = tg u_2)$. In order to improve accuracy according to the requirements of interstate standard: $\Delta B \leq 0,0001''$, the segment $[T_1, T_2]$ was reduced by two methods from left side. In order to develop non-iterative algorithm, the segment $[T_3, T_2]$ was selected where the latitude was determined not only at $H > 0$, but also at $H < 0$. Using auxiliary variables, the general secant equation was converted into convenient form for calculations of latitude. Its error has been estimated which was analyzed for extremum. According to the proposed algorithm the latitude is determined with the error of $\Delta B \leq 1'' \cdot 10^{-10}$. This method can be applied for development of non-iterative algorithms in other segments of root isolating with subsequent comparative analysis.

Index Terms: algorithms, ellipsoid, equation errors, geodesic and orthogonal spatial coordinates, latitude, secant method.

I. INTRODUCTION

Determination of relative position of points on Earth surface and in near-earth region is a global task. Under the influence of scientific and engineering advance, solutions of this problem vary though the task is still urgent and important. The main systems of spatial coordinates for all points of Earth surface are the orthogonal coordinates X, Y, Z and the curvilinear coordinates comprised of latitude B , longitude L , and altitude H . The interrelation between these coordinates is expressed by the equations described in [1, p.6]:

$$\begin{cases} X = (N + H) \cos B \cos L, \\ Y = (N + H) \cos B \sin L, \\ Z = (N(1 - e^2) + H) \sin B, \end{cases} \quad (1)$$

which make it possible to determine the orthogonal coordinates X, Y, Z of point Q on the basis of predefined geodesic coordinates B, L, H , where $N = a/\sqrt{1 - e^2 \sin^2 B}$ is the curvature radius of the first vertical; a is the major semi-axis; $e = \sqrt{a^2 - b^2}/a$ is the first ellipsoid eccentricity; b is the minor semi-axis. Conversion reverse to (1) is expressed by more complicated dependences. Numerous publications are devoted to solution of this problem up till now [2]-[6], and others. In this regard it became necessary to carry out specialized theoretical and procedural studies on analysis and systematization of the developed methods of coordinate conversion, substantiation of development of optimum high precious algorithms. Results of such studies are defining not only for recommendations of equations but also during development of state standards.

II. METHODS

Regarding the three sought variables, B, L, H , the most simple solution:

$$tg L = Y/X, X \neq 0, \quad (2)$$

is for the geodesic longitude L , belonging to the region $0 \leq L \leq 2\pi$. Incomplete solutions of (2) are described in [7]-[10], etc. The algorithms of longitude determination are analyzed in [11]. The most significant difficulties arise upon development of algorithm for calculation of geodesic latitude on the basis of transcend equation [12, p. 192]:

$$tg B = (Z + e^2 N \sin B)/R, R = \sqrt{X^2 + Y^2} \neq 0. \quad (3)$$

The function $tg B$, hence, (3), are meaningless at $B = \pm 90^\circ$. In this case it follows from (1): $X = 0; Y = 0; Z = \pm(b + H)$; hence, the point $Q(0; 0; Z)$ is located on the ellipsoid rotation axis, $R = 0$ and the problem is solved according to the algorithm:

$$B = \frac{\pi}{2} \operatorname{sign} Z; H = |Z| - b; L - \text{any value; in particular, } L = 0.$$

In general case the equations for calculation of latitude B are derived by expansion of the function in a series [13]-[15]; by solution of (3) by iterations [1], [16]-[20] and other methods [21].

Equation (3) with the geodesic latitude B by substitution

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$$tgu = \sqrt{1 - e^2} t g B \quad (4)$$

is transformed to the equation with reduced latitude u :

$$tgu = A + C \sin u; A = \frac{Z\sqrt{1-e^2}}{R}; C = \frac{ae^2}{R}, \quad (5)$$

with less complicated structure which simplifies not only development of algorithms but their analysis as well. In order to determine the root of (5) or, which is the same, of the function

$$f(t) = t - A - C \sin u; t = tgu; \sin u = t/\sqrt{1+t^2}, \quad (6)$$

by numeric methods, it is necessary to isolate the root, that is, to determine the minimum segment where the root is located. While solving (6) by iterations, the following variables are taken as initial approximation:

$$t_0 = T_1 = A = \frac{Z\sqrt{1-e^2}}{R}; t_0 = T_2 = \frac{Z}{R\sqrt{1-e^2}}, \quad (7)$$

where T_2 is determined by (1) at $H = 0$.

T_1 and T_2 generate a segment of very low length. Indeed, using binomial expansion we obtain $T_{1,2} \approx \frac{Z}{R}(1 \pm e^2/2)$. At any Earth ellipsoid $e^2/2 < 0,004$. Hence, T_1 and T_2 after rounding off will be characterized by at least two equal primary significant digits. This property will be valid for all other points included between them which could be considered approximately equal to T_1 or T_2 with the mentioned error. This will be considered upon calculation of approximate values of the functions and error estimations.

Assuming $e = 0$ in respective equations, we will have for approximate calculations:

$$N \approx a; B \approx u; R = (a + H) \cos u; Z = (a + H) \sin u. \quad (8)$$

Taking into account (8), we define the values of $f(t)$ [22]:

$$f(T_1) \approx -\frac{ae^2}{a+H} t_0; f(T_2) \approx \frac{He^2}{a+H} t_0, \quad (9)$$

where t_0 is any value in the segment $[T_1, T_2]$.

However, $f(t)$ in this segment will have the root \bar{t} provided that $f(T_1) \cdot f(T_2) < 0$, which is possible, as follows from (9), only at $H > 0$. This root \bar{t} is unique in $[T_1, T_2]$, since the first derivative

$$f'(t) = 1 - C \cos^3 u \approx 1 - \frac{ae^2}{a+H} \cos^2 u \quad (10)$$

at $H > -a + ae^2$ will be positive.

Equation (6) by means of trigonometric identities $\sin u = t \cos u = \sin u (\sin^2 u + \cos^2 u)$ is converted as follows:

$$f(t) = t - A - C \sin u = t - A - Ct \cos u = t - A - C(\sin^3 u + t \cos^3 u),$$

its roots are determined as follows:

$$t = \phi(t) = A + C \sin u, \quad (11)$$

$$t = \phi_1(t) = A/(1 - C \cos u), \quad (12)$$

$$t = \phi_2(t) = \frac{A+C \sin^3 u}{1-C \cos^3 u}. \quad (13)$$

The studies of iterations $t_k = \phi(t_{k-1})$, where $k = 1(1)\infty$, by (11–13) are described in [21], [22].

Root isolating allowed not only to apply the developed

general convergence condition and estimations of iterations, but also to apply the secant method, one of long standing and efficient methods of approximate solution of equations with numerical coefficients. This method is also known as the method of chords, of proportional parts, of false position, of linear interpolation and in the segment $[t_0, t_1]$ is determined by the following equation [23]

$$tgu = t_0 - \frac{(t_0 - t_1)f(t_0)}{f(t_0) - f(t_1)}. \quad (14)$$

The studies in [24] were carried out using (14) and iterations with initial approximation $t_0 = T_1$ and permanent point $t_1 = T_2$.

However, for theoretical studies and solution of application tasks non-iterative algorithms are more efficient. Their application decreases the amount of calculations with approximate values; they directly provide the final result; analysis of algorithms is simplified significantly. The studies [21] have established that the highest convergence rates of iterations are achieved using (13) and (14). Hence, they could be applied for development of non-iterative algorithms of latitude prediction.

III. RESULTS AND DISCUSSION

If The error for the root \bar{t} calculated by secant method in the segment $[t_0, t_1]$ for any function $f(t)$ is determined as follows [23, p. 204]:

$$\varepsilon = -\frac{1}{2} \frac{f''(\bar{t})}{f'(\bar{t})} \varepsilon_0 \cdot \varepsilon_1, \quad (15)$$

where $\varepsilon = t - \bar{t}$; $\varepsilon_0 = t_0 - \bar{t}$; $\varepsilon_1 = t_1 - \bar{t}$.

For (6) according to the Lagrange theorem about mean value in the segment $[t_0, \bar{t}]$ we have [20]: $\varepsilon_0 = f(t_0)/f'(\bar{t})$. It follows from (10) with the error of about e^2 that $f'(\bar{t}) = 1$. Thus, with the accepted error $\varepsilon_0 = f(t_0)$. Similarly, $\varepsilon_1 = f(t_1)$, and (15) is transformed as follows:

$$\varepsilon = -\frac{1}{2} f''(\bar{t}) f(t_0) \cdot f(t_1). \quad (16)$$

As a consequence of low length of the segment $[T_1, T_2]$, it is possible to assume that t_0, t_1, \bar{t} are equal to any value $t \in [T_1, T_2]$. Then,

$$f''(\bar{t}) = f''(t) = 3Ct \cos^5 u \approx \frac{3ae^2}{a+H} \sin u \cos^3 u. \quad (17)$$

However, the errors of the function $t = tgu$ and argument u are related by the function $\Delta u = \varepsilon \cos^2 u$, with consideration for (16), (17) it can be written as follows:

$$\Delta u = -\frac{3}{2} \frac{ae^2}{a+H} \sin u \cos^5 u \cdot f(t_0) \cdot f(t_1). \quad (18)$$

If in (14) to assume $t_0 = T_1$, and $t_1 = T_2$, then (18) with consideration for (9) is transformed as follows:

$$\Delta u = \frac{3}{2} \frac{a^2 H e^6}{(a+H)^3} \sin^3 u \cos^3 u = \frac{3}{16} \frac{a^2 H e^6}{(a+H)^3} (\sin 2u)^3.$$

The same result is obtained from the general iteration error Δu_k [12] at $k = 1$, its highest value $\Delta u = \Delta B = 0,0017''$ is achieved at $B = 45^\circ$ and $H = a/2$.



Equation (13) is identical [21, p. 41] to application of the Newton method (tangent) to solution of (6): $t = t - f(t)/f'(t)$. Its error is determined by the equation [23, p. 196] $\varepsilon = \frac{1}{2} \frac{f''(\bar{t})}{f'(\bar{t})} \varepsilon_0^2$.

Using previously obtained regularities we have:

$$\Delta u = \frac{3}{2} f^2(t_0) \frac{ae^2}{a+H} \sin u \cos^5 u. \quad (19)$$

The Boring equation derived by expansion into series [13] follows from (13) at $t = T_2$ using (4). Its error is determined by (19) at $t_0 = T_2$:

$$\Delta u = \frac{3}{2} \frac{aH^2 e^6}{(a+H)^3} \sin^3 u \cos^3 u,$$

the maximum of which $\Delta u = 0,0017''$ is at $B = 45^\circ$, $H = 2a$. Therefore, on the basis of (13) and (14) the latitude is calculated with the maximum error equaling to $0.0017''$. According to the requirements of interstate standard [8], the latitude error should not exceed $\Delta B \leq 0.0001''$. In order to achieve such accuracy on the basis of (13) or (14), it is required to decrease the segment of root isolating $[T_1, T_2]$.

In the secant method the root \bar{t} is approached from the end of T_1 , and T_2 remains the same [21, p. 39]. The segment $[T_1, T_2]$ can be decreased from the left by (11) and (12).

Assuming $Z > 0$ and taking the root $t_0 = T_1$ as initial value after the first iteration by (11), we have as follows:

$$T_5 = tgu_5 = A + CT_1/\sqrt{1+T_1^2} = \frac{Z\sqrt{1-e^2}}{R} (1 + ae^2/b_0);$$

$$b_0 = R\sqrt{1+T_1^2}.$$

Since the iteration by (11) is converged to \bar{t} monotonously starting from T_1 [22, Part 1], then the obtained T_5 will be closer to the root \bar{t} than T_1 , that is, $T_1 > T_5 > \bar{t}$. In this regard the root \bar{t} will be in the segment $T_5 < \bar{t} < T_2$.

Taking into account (8), the approximate value is $f(T_5) = -\left(\frac{ae^2}{a+H}\right)^2 t \cos^2 u$. Comparing $f(T_5)$ and $f(T_2)$ we conclude that the root \bar{t} will belong to the segment $[T_5, T_2]$ only at $H > 0$. Similarly, using substitution $t_0 = T_2$ after the first iteration by (12), we obtain:

$$T_3 = tgu_3 = \frac{A}{1-c/\sqrt{1+T_2^2}} = \frac{Z\sqrt{1-e^2}}{R(1-ae^2/a_0)}; a_0 = R\sqrt{1+T_2^2}.$$

The root by (12) is approached from both sides [22, Part 1], thus, T_3 will be located on the opposite side of T_2 , that is, $T_3 < \bar{t} < T_2$. If $Z < 0$, then, to the contrary, $T_2 < \bar{t} < T_3$, with the error of about e^6

$$f(T_3) = -\frac{aHe^4}{(a+H)^2} t \sin^2 u. \quad (20)$$

It follows from $f(T_2)$ and $f(T_3)$ that the function $f(t)$ will possess the values of opposite signs ($f(T_2) \cdot f(T_3) < 0$) not only at $H > 0$, but in overall region of the iteration convergence $-a + ae^2 < H < +\infty$. This result substantiates the relevance of studying the Boring equation at $H = -10km$ [25].

Assuming $t_0 = T_2$, $t_1 = T_3$ in (14), we obtain the secant formula for calculation of reduced latitude:

$$tgu = T_2 - \frac{(T_2 - T_3)f(T_2)}{f(T_2) - f(T_3)}. \quad (21)$$

Its error is determined according to (18) with consideration for (9), (10) at $f(t_0) = f(T_2)$, $f(t_1) = f(T_3)$:

$$\Delta u = \frac{3}{2} \frac{a^2 H^2 e^8}{(a+H)^4} \sin^5 u \cos^3 u; a + H \neq 0. \quad (22)$$

Analyzing optimums of two functions $y_1 = \sin^5 u \cos^3 u$ and $y_2 = \frac{a^2 H^2}{(a+H)^4}$ contained in (22), we obtain as follows:

the function y_1 obtains its maximum value $y_1 = \frac{75\sqrt{15}}{4096} \approx 0,0709$ at $u = 52^\circ 14'$; the function $y_2(H)$ in the variation range has critical points $H = 0$ and $H = a$, where $y_2(0) = 0$; $y_2(a) = \frac{1}{16}$. Hence, at $H = 0 \Delta u = 0$, and at $H = a \Delta u = 2.8'' \cdot 10^{-6}$. However, upon latitude determination by any method using (3) or (5), the error $\Delta B \approx \Delta u$ will increase when $a + H \rightarrow 0$. For instance, at $H = -a/2$ using (22) we obtain $\Delta u = 1.8'' \cdot 10^{-5}$ and at $H = -a + ae^2 \Delta u \approx 6^0$.

For the points of Earth surface at $|H| \leq 10km$, highly precious results are obtained at the error of $\Delta u \approx \Delta B \leq 1'' \cdot 10^{-10}$. In order to calculate by (21), let us introduce auxiliary values which simplify calculations and algorithm structure. Let us present T_3 in the form of $T_3 = c_0 T_2$, where $c_0 = (1 - e^2) / \left(1 - \frac{ae^2}{a_0}\right)$; $a_0 = R\sqrt{1+T_2^2}$; $T_2 = \frac{Z}{R\sqrt{1-e^2}}$.

Then, $T_2 - T_3 = (1 - c_0)T_2$; $f(T_2) = e^2 T_2 - C \sin u_2$; $f(T_2) - f(T_3) = (1 - c_0)T_2 + C(\sin u_3 - \sin u_2)$.

With these expressions (21) is transformed as follows:

$$tgu = T_2 \left(1 - \frac{1 - c_1 \sin u_2}{k_2 + c_2(\sin u_3 - \sin u_2)}\right). \quad (23)$$

During calculations by (23) let us use constants obtained by the ellipsoid parameters with major semi-axis α and denominator f of compression $\alpha = (a - b)/b = 1/f$:

- 1) $k_0 = \sqrt{1 - e^2} = (f - 1)/f$
- 3) $k_1 = ae^2 = a(2f - 1)/f^2$
- 2) $k_2 = 1/e^2 = f^2/(2f - 1)$
- 4) $b = k_0 \cdot a$

Solution according to (23) is carried as follows:

- 1) $R = \sqrt{X^2 + Y^2} \neq 0$;
- 2) $T_2 = Z/(k_0 R)$;
- 3) $a_0 = R\sqrt{1+T_2^2}$;
- 4) $c_0 = \frac{k_0^2}{1 - k_1/a_0}$;
- 5) $c_1 = b/Z$;
- 6) $c_2 = c_1/(1 - c_0)$;
- 7) $\sin u_2 = \sin(\arctg T_2)$;
- 8) $\sin u_3 = \sin(\arctg(c_0 T_2))$;
- 9) Using (23) we determine tgu ;
- 10) Geodesic latitude $B = \arctg\left(\frac{tgu}{k_0}\right)$ is calculated, its error is $\Delta B = \Delta u$.

Reliability of the proposed algorithm was verified by reciprocal transformation using (1) and (23) with subsequent comparison using (22).

IV. CONCLUSION

1. Using the non-iterative algorithm (23), the latitude B at $H \geq 0$ is calculated with the maximum error $\Delta B = 2,8'' \cdot 10^{-6}$. For the earth surface points at $|H| \leq 10km$ the obtained results are precise with the error of $\Delta B \leq 1'' \cdot 10^{-10}$.



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2. At negative altitudes, when $a + H \rightarrow 0$, the error ΔB increases and at $H = -a + ae^2$ equals to $\Delta B \approx 6^\circ$. In extreme case $R = 0, Z \neq 0$ the equation is solved as follows:

$$B = \frac{\pi}{2} \operatorname{sign} Z; H = |Z| - b; L - \text{any value.}$$

3. The proposed secant method can be used for development of non-iterative algorithms in other segments of root isolating and then to perform comparative analysis.

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