Separation Axioms in N-ARY Topology

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Abstract: In the year 2011, Nithyanantha Jothi and Thangavelu introduced the concept of binary topology. Recently the authors extended the notion of binary topology to n-ary topology where n>1 an integer and extended some concepts in general topology to n-ary topology. In this paper separation axioms are introduced in n-ary spaces and their basic properties are discussed.

Keywords: n-ary topology, n-ary open, n-ary closed and n-ary continuity.

I. INTRODUCTION

Nithyanantha Jothi and Thangavelu[5-10] studied the notion of a binary topology and investigated the corresponding closure and interior operators in binary topological spaces. Following this topologists extended this concept to soft topological, generalized topological and supra topological spaces[11-14]. The second and the first authors[11,12,13] discussed nearly open sets in binary topology and n-ary topology. They also discussed n-ary closed sets[14], n-ary subspaces [15] and n-ary continuity[16]. In general topology, separation axioms play a vital role to classify topological spaces and will be used to study some topological properties. In this paper, by introducing new axioms in n-ary topological spaces, we will investigate the properties of n-ary topologies.

II. PRELIMINARIES

Let \( X_1, X_2, \ldots, X_n \) be the nonempty sets and \( P(X_1) \times P(X_2) \times \cdots \times P(X_n) \) be the Cartesian product of the their power sets. Examples can be constructed to show that the two notions ‘product of power sets’ and ‘power set of the products’ are different. Any typical element in \( P(X_1) \times P(X_2) \times \cdots \times P(X_n) \) is of the form \( A=(A_1, A_2, \ldots, A_n) \) where \( A_i \subseteq X_i \) for \( i \in \{ 1, 2, 3, \ldots, n \} \).

Throughout this paper \( A=(A_1, A_2, \ldots, A_n) \) and \( B=(B_1, B_2, \ldots, B_n) \) are members in \( P(X_1) \times P(X_2) \times \cdots \times P(X_n) \). The notations and terminologies that are used here are taken from [11-16]. The union, intersection and complementation of n-ary elements are defined component wise. Let \( T \subseteq P(X_1) \times P(X_2) \times \cdots \times P(X_n) \).

Definition 2.1:

\( T \) is an n-ary topology on \( (X_1, X_2, \ldots, X_n) \) if the following axioms are satisfied.

(i) \( (\emptyset, \emptyset, \ldots, \emptyset) = \emptyset \in T \)

(ii) \( (\emptyset, X_2, \ldots, X_n) = X \in T \)

(iii) \( T \) is closed under finite intersections

(iv). If \( (A_1, A_2, \ldots, A_n) \in T \) for each \( \alpha \in \Omega \) then \( \bigcup_{\alpha \in \Omega} (A_1\alpha, A_2\alpha, \ldots, A_n\alpha) \in T \).

If \( T \) is an n-ary topology then the pair \( (X, T) \) is an n-ary topological space. The element \( x=(x_1, x_2, \ldots, x_n) \in X \) is an n-ary point of \( (X, T) \) and the members \( A \) of \( P(X_1) \times P(X_2) \times \cdots \times P(X_n) \) are called the n-ary sets. The members of \( T \) are called the n-ary open sets in \( (X, T) \). It is noteworthy to see that product topology on \( X_1 \times X_2 \times \cdots \times X_n \) and n-ary topology on \( (X_1, X_2, \ldots, X_n) \) are independent concepts as any open set in product topology is a subset of \( X_1 \times X_2 \times \cdots \times X_n \) and an open set in an n-ary topology is a member of \( P(X_1) \times P(X_2) \times \cdots \times P(X_n) \).

Lemma 2.2:

\( T_1 = \{ A_1; (A_1, A_2, \ldots, A_n) \in T \} \) is a topology on \( X_1 \).

Lemma 2.3:

Let \( \tau_i \) be a topology on \( X_i \). Then \( T=\tau_1 \times \tau_2 \times \cdots \times \tau_n = \{(A_1, A_2, \ldots, A_n); A_i \in \tau_i \} \) is an n-ary topology on \( X \). Moreover \( T_1 = \tau_1 \).

Definition 2.4:

\( (A_1, A_2, \ldots, A_n) \) is n-ary closed if \( (A_1, A_2, \ldots, A_n)^c = (X \setminus A_1, X \setminus A_2, \ldots, X \setminus A_n) = X \setminus A \) is n-ary open in \( (X, T) \).

For the notations and terminologies that are used here the reader may consult [11 -16].

III. HAUSDORFF AXIOM IN n-ARY TOPOLOGY

Definition 3.1:

An n-ary space \( (X, T) \) satisfies an n-ary T2-axiom if for any two absolutely distinct n-ary elements \( x, y \) of \( X \), there are n-ary disjoint open sets \( U \) and \( V \) with \( x \in U \) and \( y \in V \) that is if \( x \neq y \Rightarrow \)there are n-ary open sets \( U \) and \( V \) such that \( U \cap V = \emptyset \), \( x \in U \), \( y \in V \).

An n-ary space satisfying n-ary T2-axiom is called an n-ary Hausdorff space.
Theorem 3.2:
Let \( x, y \in X \). The following are equivalent.
(i). \( X \) is n-ary Hausdorff
(ii). If \( y \neq x \), there is an n-ary nbd \( U(x) \) such that \( y \notin n-CI(U(x)) \).

Proof:
Suppose \( X \) is n-ary Hausdorff and \( y \neq x \). Then, there are n-ary disjoint open sets \( U(x) \) and \( V(y) \) such that \( x \notin U \) and \( y \notin V \). Now, \( U \cap V = \emptyset \) implies \( y \notin n-CI(U(x)) \). This proves (i) \( \Rightarrow \) (ii).

Theorem 3.3:
(i). The n-ary Hausdorff topologies are invariant under n-ary closed bijections.
(ii). An n-ary Hausdorff space is n-ary Hausdorff.

Definition 4.1:
An n-ary Hausdorff space \((X,T)\) is said to satisfy an n-ary \( T_3 \)-axiom if for any n-ary element \( x \) of \( X \) and for every n-ary closed set \( F \) with \( x \in F \), there are n-ary disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( V \subseteq F \).

An n-ary space satisfying n-ary \( T_3 \)-axiom is called an n-ary regular space.

Theorem 4.2:
Let \((X, T)\) be an n-ary Hausdorff and \( x \in X \). The following are equivalent.
(i). \( X \) is n-ary \( T_3 \)
(ii). For an n-ary open nbd \( U \) of \( x \), there is an n-ary open \( (x) \) with \( x \in V \subseteq n-CI(U) \).
(iii). For each n-ary closed set \( A \) with \( x \in A \), there is an n-ary open nbd \( V \) of \( x \) such that \( n-CI(V) \supseteq A \).

Proof:
Suppose \((X, T)\) is n-ary \( T_3 \). Let \( U \) be an n-ary open nbd of \( x \). Then \( x \neq B \). Since \((X, T)\) is n-ary \( T_3 \), there are n-ary disjoint open sets \( V \) and \( W \) such that \( x \in V \) and \( B \subseteq W \). Since \( V \cap W = \emptyset \) follows from the fact that \( x \notin V \subseteq n-CI(V) \subseteq X \), \( U \subseteq W \). This proves (i) \( \Rightarrow \) (ii).

V. NORMAL AXIOM IN n-ARY TOPOLOGY

Definition 5.1:
An n-ary Hausdorff space \((X,T)\) is known to satisfy an n-ary \( T_4 \)-axiom if for the given disjoint n-ary closed sets \( A \) and \( B \) in \((X,T)\), there are n-ary disjoint open sets \( G \) and \( H \) such that \( A \subseteq G \) and \( B \subseteq H \). An n-ary space satisfying n-ary \( T_4 \)-axiom is called an n-ary normal space.

Theorem 5.2:
The following are equivalent.
(i). \( X \) is n-ary \( T_4 \)
(ii). If \( A \) is an n-ary closed set in \( X \) and \( U \) is an n-ary open set \( U \supseteq A \), there is an n-ary open set \( V \) such that \( A \subseteq V \subseteq n-CI(V) \subseteq U \).
(iii). If \( A \) and \( B \) are disjoint n-ary closed sets then \( U \) is an n-ary open set \( U \) with \( A \supseteq U \) and \( (n-CI(U) \cap B) = \emptyset \).
Proof: Suppose \((X, T)\) is n-ary \(T_2\). Let \(A \subseteq U\) be an \(n\)-ary set and \(U\) be an \(n\)-ary open set in \(X\) so that \(X \cap U\) and \(A\) are disjoint \(n\)-ary closed sets in \(X\) that implies there are \(n\)-ary disjoint open sets \(V\) and \(W\) such that \(V \subseteq X \setminus W \subseteq X \setminus U\). This proves (i) \(\Rightarrow\) (ii).

Now suppose (ii) holds. Let \(A\) and \(B\) be any two disjoint \(n\)-ary closed sets in \(X\). Then by applying (ii), there is an \(n\)-ary open set \(U\) such that \(A \subseteq U \subseteq n\-\text{CI} \ U \subseteq W\) that implies \(B \subseteq X \setminus (n\-\text{CI} \ U)\) so that 

\[
(n\-\text{CI} \ U) \cap B = \Phi.
\]

This proves (ii) \(\Rightarrow\) (iii).

If (iii) holds and if \(t\) and \(B\) be the two disjoint \(n\)-ary closed sets in \(X\) then 

\[
(n\-\text{CI} \ U) \cap B = \Phi
\]

that shows that \(X \cap (n\-\text{CI} \ U)\) and \(U\) and \(V\) are the expected \(n\)-ary open sets. This proves (iii) \(\Rightarrow\) (i).

**Theorem 5.3:**

(i). The \(n\)-ary normal topologies are invariant under \(n\)-ary continuous open injection

(ii). The \(n\)-ary normal topologies are invariant under \(n\)-ary continuous closed surjections.

(iii). An \(n\)-ary closed subspace of the \(n\)-ary normal space is \(n\)-ary normal

(iv). \((X, T)\) is \(n\)-ary normal iff each \((X_i, T_i)\) is normal.

**Proof:** Let \(f: X \rightarrow Y\) be an \(n\)-ary continuous. Let \(A\) and \(B\) be any two disjoint \(n\)-ary closed sets in \(Y\). If \(f\) is a surjection and \((X, T_X)\) is \(n\)-ary normal then \(f(A)\) and \(f(B)\) are the disjoint \(n\)-ary closed sets in \(X\) that implies there are disjoint \(n\)-ary open sets \(U\) and \(V\) in \(X\) such that \(f(A) \subseteq U\) and \(f(B) \subseteq V\) so that \(f(U)\) and \(f(V)\) are the desired \(n\)-ary open sets in \(Y\). This proves (i). If further \(f: X \rightarrow Y\) is \(n\)-ary closed also and if \((Y, T_Y)\) is \(n\)-ary normal then \(f(A)\) and \(f(B)\) are disjoint \(n\)-ary closed sets in \(Y\) so that there are disjoint \(n\)-ary open sets \(U\) and \(V\) in \(Y\) with \(f(A) \subseteq U\) and \(f(B) \subseteq V\) that implies, \(f(U)\) and \(f(V)\) are the desired disjoint \(n\)-ary open sets in \(X\). This proves (ii). Let \((Y, T_Y)\) be an \(n\)-ary closed subspace of an \(n\)-ary normal space \((X, T_X)\). If \(A \cap Y\) and \(B \cap Y\) are any two disjoint \(n\)-ary closed sets in \(Y\) then \(A \cap Y\) and \(B \cap Y\) are \(n\)-ary closed in \(X\) so that there are disjoint \(n\)-ary open sets \(U\) and \(V\) in \(X\) with \(A \cap Y \subseteq U\) and \(B \cap Y \subseteq V\) that implies \(A \cap Y \subseteq U\) and \(B \cap Y \subseteq V\) are the desired \(n\)-ary open sets in \(Y\). This shows (iii). Now let \((X, T)\) be \(n\)-ary normal Fix \(j\). Let \(P_j\) and \(Q_j\) be any two disjoint closed sets in \(X_j\). Choose \(n\)-ary closed sets \(A\) and \(B\) with \(A \subseteq P_j\) and \(B \subseteq Q_j\). Since \((X, T)\) is \(n\)-ary normal there are disjoint \(n\)-ary open sets \(U\) and \(V\) with \(A \subseteq U\) and \(B \subseteq V\) so that \(P_j \subseteq U\) and \(Q_j \subseteq V\) Therefore \(U_j\) and \(V_j\) are the desired open sets in \((X_j, T_j)\) that implies \((X_j, T_j)\) is normal. The converse part is obvious. This proves (iv).

**VI. CONCLUSION**

The notions of Hausdorff, regular and normal separation axioms are extended to \(n\)-ary topological spaces. These axioms are investigated in \(n\)-ary topological settings. Further it has been established that an \(n\)-ary space is \(n\)-ary Hausdorff, \(n\)-ary regular and \(n\)-ary normal iff the corresponding topologies on the coordinate spaces are Hausdorff, regular and normal respectively.

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