On Harmonic Index of new Operation of Graphs

S.Uma, N. Velmurugan

ABSTRACT--- The harmonic index of a graph Γ, denoted by H(Γ), is defined as $H(\Gamma)=\sum_{rs\in E(\Gamma)}\frac{2}{Dg\Gamma(r)+Dg\Gamma(s)}$ ***where*** $Dg\Gamma(r)$ ***is the drgree of a vertex r in Γ. In this paper , we concentrate the upper bounds of the harmonic index for four types of derived graphs.***

Keywords: Harmonic index, join, subdivision, total graph.
MSC: 05C12, 05C76.

I. INTRODUCTION

A *topological index*of a simple connected graphs is a numerical quantity related to the graph and its does not depend on the labeling or pictorial representation of the graph. It has been applied for studying quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR) extensively in which the biological activity or other properties of molecules are correlated with their chemical structures, see [6]. In this present literature, large number of graph based structure indices have been put forward, that all depend only on the degree of the vertices of the graph. For more details on degree based topological index and on their comparative study can be found in [7, 8, 9, 10, 4, 5].

In 1986, Gutman et al, proposed the first degree based index called Zagreb index which is

Defined following may

$M_1(\Gamma) = \sum_{rs\in E\Gamma}\big(Dg\Gamma(r) + Dg\Gamma(s)\big) and M_2(\Gamma) =$ $rs\in E\Gamma Dg\Gamma r.Dg\Gamma s$, Where $Dg\Gamma(s)$ is a degree of a vertex s in Γ.For a simple connected graph Γ, the harmonic index denoted by H(Γ) is defined as $H(\Gamma)=\sum_{rs\in E(\Gamma)}\frac{2}{Dg\Gamma(r)+Dg\Gamma(s)}$.

Deng et al. [18] studied the relation between the harmonic index of a connected graph Γand its chromatic number $\chi(\Gamma)$. Zhong [11, 12, 13] obtained the maximum and minimum values of the harmonic index for several graphs such as tree, unicyclic graph and graph with girth at least $k(k \geq 3)$. Further they characterized the harmonic index for corresponding extremal graphs. The comparison between the harmonic index of a connected graphΓ and its matching numbers are established by Lv et al. [16,17]. Shwetha et al. [15] obtained the formulae for the harmonic index of some derived graphs. Since graph operation plays an important role to study the infinity graphs which are derived from the smaller graphs, in this connection, we present several upper bounds for the harmonic index of the $VJ-$ join derived graphs.

II. PRELIMINARIES

For a simple connected graphΓ, we obtain the following derived graphs.

i. The*subdivisiongraph*of a graph G denoted by $S\,(\Gamma)$ is the graph obtained fromΓby replacing each edges ofΓby a path of length two.
ii. $R(\Gamma)$is the graph obtained fromΓby adding a new vertex corresponding to each edge ofΓ,then joining each new vertex to the end vertices of the corresponding edge.
iii. Q(Γ) is a graph obtained fromΓby inserting a new vertex into each edges ofΓ, then joining with edges those pairs of new vertices on adjacent edges ofΓ.
iv. The*total graph*of a graph*Γ* denoted by T(Γ) has as its vertices the edges and vertices of Γ. Adjacency in T(Γ) is defined as adjacency or incidence for the corresponding elements ofΓ.

The *join* of two simple connected graphs Γ_1 *and* $\Gamma_2$denoted by$\Gamma_1 \vee \Gamma_2$, is a graph with vertex set $V(\Gamma_1)\cup V(\Gamma_2)$ and every vertex of Γ_1 is join to all the vertex of Γ_2.

Eliasi and Taeri .[1] proposed the four new operations of the graph based on the Cartesian product. Sarala et al .[2] studied the new operations of the graphs based on the composition. In this sequence, Sarkar et al. [3] discussed another type of graph operation based on join as follows.

Let $J = \{S, R, Q, T\}$ and $I(\Gamma)$ denotes the set of vertices $J(\Gamma)$ which are inserted into each edge ofΓ, so that $V\big(J(\Gamma)\big) = V(\Gamma) \cup I(\Gamma)$.The $VJ - join\ graph$ $\Gamma_1 \vee_s \Gamma_2$ of Γ_1 *and* Γ_2 is a graph with vertex set $V\big(J(\Gamma_1)\big)\cup V\big((\Gamma_2)\big)$ and edge set $E(\Gamma_1)\cup E(\Gamma_2)\cup\{rs|r \in V(\Gamma_1), s \in V(\Gamma_2)\}$ obtained from $J(\Gamma_1) and\ \Gamma_2$ by joining each vertex of Γ_1 with every vertex of Γ_2.

Lemma 2.1: for two non negative real numbers x and yThen $\frac{1}{x+y} \leq \frac{1}{4x} + \frac{1}{4y}$. equality holds if and only if x=y.

VS-join graph

Let $\Gamma_K, K = 1{,}2$ be

two different graphs. The *VS-join graph* denoted by $\Gamma_1 \vee_s \Gamma_2$, isa graph obtained from the subdivision graph

$S(\Gamma_1)\ and\ (\Gamma_2)$ by joining

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S.Uma,Department of Mathematics, Government Arts college for (Women), Mayiladuthurai, TN, India.
N. Velmurugan, Department of Mathematics, TheivannaiAmmal College for Women(Autonomous), Villupuram, TN, India.

each vertex of $V(\Gamma_1)$ to all vertex of (Γ_2). By using the structure of the graph $\Gamma_1 \vee_s \Gamma_2$. we can easily get the proof of the following lemma.

Lemma:3.1

For the connected graph $\Gamma_k, k = 1,2$, the degree of a vertex r in $\Gamma_1 \vee_s \Gamma_2$ is

$$D_g\Gamma_1 \vee_s \Gamma_2(r) = \begin{cases} D_{g\Gamma_1}(r) + |V(\Gamma_2)|, if r \in V(\Gamma_1) \\ D_{g\Gamma_2}(r) + |V(\Gamma_1)|, if r \in V(\Gamma_2) \\ 2, if r \in I(\Gamma_1) \end{cases}$$

Now we find the upper bound for harmonic index of the graphs $\Gamma_1 \vee_s \Gamma_2$

Theorem 3.2 For the simple connected graphΓ_k with p_k vertices and q_k edges k=1,2, $H(\Gamma_1 \vee_s \Gamma_2) \leq \frac{H(\Gamma_2)}{4} + \frac{p_2 ID(\Gamma_1)+p_1 ID(\Gamma 2)}{8} + \frac{p_1 p_2}{2(p_1+p_2)} + \frac{p_1}{2} + \frac{q_1}{p_2+2} + \frac{q_2}{4p_1}$.

Proof: By using the definition of harmonic index, we get

$$H(\Gamma_1 \vee_s \Gamma_2) = \sum_{rs\in E(\Gamma_1 \vee_s \Gamma_2)} \frac{2}{Dg_{\Gamma_1 \vee_s \Gamma 2}(r) + Dg_{\Gamma_1 \vee_s \Gamma_2}(s)}.$$

We can splitted up $E(\Gamma_1 \vee_s \Gamma_2)$ into three subsets , namely,

$E_1 = \{rs \in E(\Gamma_1 \vee_s \Gamma_2)/rs \in E(S(\Gamma_1))\}$,

$E_2 = \{rs \in E(\Gamma_1 \vee_s \Gamma_2)/rs \in E((\Gamma_2)\}$ and

$E_3 = \{rs \in E(\Gamma_1 \vee_s \Gamma_2)/r \in V(\Gamma_1), s \in V(\Gamma_2)\}$.

Contribution of the edges in E_1 to the harmonic index of $\Gamma_1 \vee_s \Gamma_2$ is given by

$$H(\Gamma_1 \vee_s \Gamma_2) = \sum_{rs\in E_1} \frac{2}{Dg_{\Gamma_1 \vee_s \Gamma 2}(r) + Dg_{\Gamma 1 \vee_s \Gamma_2}(s)}.$$

$=\sum_{rs\in S(\Gamma_1)} \frac{2}{Dg\Gamma_1(s)+p_2+2}$., by lemma 3.1

$$= \sum_{s \in V(\Gamma_1)} \frac{2Dg\Gamma_1(s)}{Dg\Gamma_1(s) + p_2 + 2}$$

$$\leq \frac{1}{4} \sum_{s \in V(\Gamma_1)} 2Dg\Gamma_1(s) \left(\frac{1}{Dg\Gamma_1(s)} + \frac{1}{p_2 + 2}\right)$$

with equality if and only if $Dg\Gamma_1(s) = p_2 + 2$.

$$= \frac{p_1}{2} + \frac{q_1}{p_2+2.} \quad (3.1)$$

Likewise, the contribution of the edges in E_2 to the harmonic index of $\Gamma_1 \vee_s \Gamma_2$is given by

$$H(\Gamma_1 \vee_s \Gamma_2) = \sum_{rs\in E_2} \frac{2}{Dg_{\Gamma_1 \vee_s \Gamma_2}(r) + Dg_{\Gamma 1 \vee_s \Gamma_2}(s)}.$$

$=\sum_{rs \in E(\Gamma_2)} \frac{2}{Dg\Gamma_2(r)+p_1+Dg\Gamma_2(s)+p_1}$ by lemma 3.1

$$\leq \frac{1}{4} \sum_{r \in E(\Gamma_2)} \sum_{s \in E(\Gamma_2)} \left(\frac{2}{Dg\Gamma_2(r) + Dg\Gamma_2(s)} + \frac{1}{p_1}\right)$$

With equality if and only if$Dg\Gamma_2(r) + Dg\Gamma_2(s) = p_1$

$$= \frac{H(\Gamma_2)}{4} + \frac{q_2}{4p_1}$$

Also for the edges in E_3, contribution to the harmonic index of$\Gamma_1 \vee_s \Gamma_2$ is given by

$H(\Gamma_1 \vee_s \Gamma_2) =$

$$\sum_{r \in V(\Gamma_1)} \sum_{s \in V(\Gamma_2)} \left(\frac{2}{Dg\Gamma_1 \vee_s \Gamma_2(r) + Dg\Gamma_1 \vee_s \Gamma_2(s)}\right)$$

$$= \sum_{r \in V(\Gamma_1)} \sum_{s \in V(\Gamma_2)} \left(\frac{2}{Dg\,\Gamma_1(r) + P_2 + Dg\,\Gamma_2(s) + P_1}\right)$$

With equality if and only if$Dg\Gamma_2(r) + Dg\Gamma_2(s) = p_1 + p_2$

$$\leq \frac{1}{8} \sum_{r\in V(\Gamma_1)} \sum_{s\in V(\Gamma_2)} \left(\frac{1}{D_{g_1}\Gamma_1(r)} + \frac{1}{D_{g_2}\Gamma_2(s)}\right) + \frac{p_1 p_2}{2(p_1 + p_2)}$$

$$= \frac{p_2 ID(\Gamma_1) + p_1 ID(\Gamma_2)}{8} + \frac{p_1 p_2}{2(p_1+p_2)} \quad (3.3)$$

The desired expression for the harmonic index of$\Gamma_1 \vee_s \Gamma_2$ is obtained by summing the above

three expression.

4. Result of theorem in VR-join graph:

The VR_join graph of Γ_1 and Γ_2, denoted by $\Gamma_1 \vee_R \Gamma_2$, is a graph obtained from $R(\Gamma_1)$ and $\Gamma_2$by joining each vertex of $V(\Gamma_1)$ to all the vertex of Γ_2. By using the structure of the graph$\Gamma_1 \vee_R \Gamma_2$, we can easily get the proof of the following lemma.

Lemma 4.1: For the connected graph $\Gamma_i, i = 1,2$, the degree of a vertex r in $\Gamma_1 \vee_R \Gamma_2$ is

$$D_g\Gamma_1 \vee_R \Gamma_2(r) = \begin{cases} D_{g_1}(r) + |V(\Gamma_2)|, if r \in V(\Gamma_1) \\ D_{g_2}(r) + |V(\Gamma_1)|, if r \in V(\Gamma_2) \\ 2, if r \in I(\Gamma_1) \end{cases}$$

Now we find the bound for harmonic index of the graph $\Gamma_1 \vee_R \Gamma_2$

Theorem 4.2: let Γ_k be a grph with p_k vertices and q_k edges, k=1,2. Then $H(\Gamma_1 \vee_R \Gamma_2) \leq \frac{H(\Gamma_1)}{4} + \frac{H(\Gamma_1)}{8} + \frac{H(\Gamma_2)}{4} + \frac{(p_2+2)I(\Gamma_{1_1})}{16} + \frac{p_1 ID(\Gamma_2)}{8} + \frac{1}{4}\left(\frac{q_1}{p_2} + \frac{q_2}{p_1}\right) + \frac{p_1{}^2 + 2p_1 p_2}{2p_2(p_1+p_2)}$.

Proof:we can splitted up $E(\Gamma_1 \vee_R \Gamma_2)$ into to four subsets, namely,

$E_1 = \{rs \in E(\Gamma_1 \vee_R \Gamma_2)/rs \in E(\Gamma_1)\}$,

$E_2 = \{rs \in E(\Gamma_1 \vee_R \Gamma_2)/rs \in E(\Gamma_2)\}$

$E_3 = \{rs \in E(\Gamma_1 \vee_R \Gamma_2)/r \in I(\Gamma_1), s \in V(\Gamma_1)\}$ and

$E_4 = \{rs \in E(\Gamma_1 \vee_R \Gamma_2)/r \in V(\Gamma_1), s \in V(\Gamma_2)\}$.

Contribution of the edges in E_1 to the harmonic index of $\Gamma_1 \vee_R \Gamma_2$ is given by

$$H(\Gamma_1 \vee_R \Gamma_2) = \sum_{rs\in E_1} \frac{2}{Dg_{\Gamma_1 \vee_R \Gamma_2}(r) + Dg_{\Gamma 1 \vee_R \Gamma_2}(s)}.$$

$=\sum_{rs \in E(\Gamma_1)} \frac{2}{2Dg\Gamma_1(r)+p_2+2Dg\Gamma_1(s)+p_2}$ by lemma 4.1

$$\leq \frac{1}{2} \sum_{r \in E(\Gamma_1)} \sum_{s \in E(\Gamma_1)} \left(\frac{1}{2Dg\Gamma_1(r) + 2Dg\Gamma_1(s)} + \frac{1}{2p_2}\right)$$

With equality if and only if$2Dg\Gamma_1(r) + 2Dg\Gamma_1(s) = 2p_2$

$$= \frac{H(\Gamma_1)}{8} + \frac{q_1}{4p_2}$$

The contribution of the edges in E_2 to the harmonic index of $\Gamma_1 \vee_R \Gamma_2$is given by

$$H(\Gamma_1 \vee_R \Gamma_2) = \sum_{rs\in E_2} \frac{2}{Dg_{\Gamma_1 \vee_{R\Gamma_2}}(r) + Dg_{\Gamma 1 \vee_R \Gamma_2}(s)}.$$

$=\sum_{rs \in E(\Gamma_2)} \frac{2}{Dg\Gamma_2(r)+p_1+Dg\Gamma_2(s)+p_1}$ by lemma 4.1

$$\leq \frac{H(\Gamma_2)}{4} + \frac{q_2}{p_1}$$

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The contribution of the edges in E_3 to the harmonic index of $\Gamma_1 \vee_R \Gamma_2$ is given by

$H(\Gamma_1 \vee_R \Gamma_2) = \sum_{r\in I(\Gamma_1)} \sum_{s\in V(\Gamma_1)} \frac{2}{Dg_{\Gamma_1\vee_R\Gamma_2}(r)+Dg_{\Gamma_1\vee_R\Gamma_2}(s)}$

$=\sum_{r\in I(\Gamma_1)} \sum_{s\in V(\Gamma_1)} \frac{2}{Dg_{I(\Gamma_1)}(r)+p_2+2Dg_{\Gamma_1}(s)}$ by lemma 4.1

$$\le \frac{1}{2}\sum_{s\in V(\Gamma_1)} \left(\frac{1}{4Dg_{\Gamma_1}(s)}+\frac{1}{p_2}\right),$$

with equality if and only if $4Dg_{\Gamma_1}(s) = p_2$

$$= \frac{ID(\Gamma_1)}{8} + \frac{p_1}{2p_2}$$

the contribution of the edges in E_4 to the harmonic index of $\Gamma_1 \vee_R \Gamma_2$ is given by

$H(\Gamma_1 \vee_R \Gamma_2) = \sum_{r\in V(\Gamma_1)} \sum_{s\in V(\Gamma_2)} \frac{2}{Dg_{\Gamma_1\vee_R\Gamma_2}(r)+Dg_{\Gamma_1\vee_R\Gamma_2}(s)}$

$=\sum_{r\in V(\Gamma_1)} \sum_{s\in V(\Gamma_2)} \frac{2}{2Dg_{\Gamma_1}(r)+p_2+Dg_{\Gamma_2}(s)+p_1}$ by lemma 4.1

$$\le \frac{p_2 ID(\Gamma_1)}{16} + \frac{p_1 ID(\Gamma_2)}{8} + \frac{p_1p_2}{2(p_1+p_2)}.$$

By summing the above four expression, the desired expression for the harmonic index of $\Gamma_1 \vee_R \Gamma_2$ is obtained.

5.Result of theorem VQ-join graph

The VQ-join graph of Γ_1 *and* Γ_2, is graphdenoted from the graph by $Q(\Gamma_1)$ andΓ_2 by joining each vertex of $V(\Gamma_1)$ to every vertex of Γ_2 and is denoted by$\Gamma_1 \vee_Q \Gamma_2$. The line graph of Γ, denoted by L(Γ) is the graph whose vertices correspond to the edges of Γ with two vertices being adjacent if and only if the corresponding edges in Γ have a vertex in common.By using the structure of the graph $\Gamma_1 \vee_Q \Gamma_2$, we can easily get the proof of the following lemma

Lemma 5.1: For the connected graph $\Gamma_i, i = 1,2$, the degree of a vertex r in $\Gamma_1 \vee_Q \Gamma_2$ is

$$Dg_{\Gamma_1\vee_Q\Gamma_2}(r) = \begin{cases} Dg_{\Gamma_1}(r) + |V(\Gamma_2)|, if r \in V(\Gamma_1) \\ Dg_{\Gamma_2}(r) + |V(\Gamma_1)|, if r \in V(\Gamma_2) \\ Dg_{\Gamma_1}(j) + Dg_{\Gamma_1}(k), if e = jk \in I(\Gamma_1) \end{cases}$$

Now we compute the upper bound for harmonic index of the graph $\Gamma_1 \vee_Q \Gamma_2$

Theorem 5.2: let Γ_k be a grph with p_k vertices and q_k edges, k=1,2. Then $H(\Gamma_1 \vee_Q \Gamma_2) \le \frac{H(L(\Gamma_1))}{4} + \frac{M_1(\Gamma_1)}{16} + \frac{H(\Gamma_1)}{8} + \frac{H(\Gamma_2)}{4} + \frac{p_2 ID(\Gamma_1)+p_1 ID(2)}{8} + \frac{1}{4}\left(\frac{q_1}{p_2}+\frac{q_2}{p_1}\right) + \frac{p_1p_2}{2(p_1+p_2)} - \frac{q_1}{8}$.

Proof: By using the definition of harmonic index, we have

$H(\Gamma_1 \vee_Q \Gamma_2) = \sum_{rs\in E(\Gamma_1\vee_Q\Gamma_2)} \frac{2}{Dg_{\Gamma_1\vee_Q\Gamma_2}(r)+Dg_{\Gamma_1\vee_Q\Gamma_2}(s)}$.

we can splitted up $E(\Gamma_1 \vee_R \Gamma_2)$ into two four subsets, namely,

$E_1 = \{rs \in E(\Gamma_1 \vee_Q \Gamma_2)/r,s \in I(\Gamma_1)\}$,

$E_2 = \{rs \in E(\Gamma_1 \vee_Q \Gamma_2)/r,s \in I(\Gamma_1)\}$

$E_3 = \{rs \in E(\Gamma_1 \vee_Q \Gamma_2)/rs \in E(\Gamma_2)$ and

$E_4 = \{rs \in E(\Gamma_1 \vee_Q \Gamma_2)/r \in V(\Gamma_1), s \in V(\Gamma_2)\}$.

Contribution of the edges in E_1 to the harmonic index of $\Gamma_1 \vee_Q \Gamma_2$ is given by

$H(\Gamma_1 \vee_Q \Gamma_2) = \sum_{r,s\in I(\Gamma_1)} \frac{2}{Dg_{\Gamma_1\vee_Q\Gamma_2}(r)+Dg_{\Gamma_1\vee_Q\Gamma_2}(s)}$.

$$= \sum_{rs\in I(\Gamma_1)} \frac{2}{Dg_{I(\Gamma_1)}(r)+Dg_{I(\Gamma_1)}(s)}, by lemma\ 5.1$$

$= \sum_{rs,st\in E(\Gamma_1)} \frac{2}{Dg_{\Gamma_1}(r)+Dg_{\Gamma_1}(t)+2Dg_{\Gamma_1}(s)}$

$$= \sum_{e=rs,f=st\in E(L(\Gamma_1))} \frac{2}{Dg_{L(\Gamma_1)}(e)+Dg_{L(\Gamma_1)}(f)+4}$$

$$\le \frac{1}{4}\sum_{e,f\in E(L(\Gamma_1))} \left(\frac{2}{Dg_{L(\Gamma_1)}(e)+Dg_{L(\Gamma_1)}(f)}+\frac{1}{2}\right)$$

$$= \frac{H(L(\Gamma_1))}{4} + \frac{|E(L(\Gamma_1))|}{8}.$$

Since $|E(L(\Gamma_1))| = \frac{M_1(\Gamma_1)}{2} - q_1$, we have

$H(\Gamma_1 \vee_Q \Gamma_2) = \frac{H(L(\Gamma_1))}{4} + \frac{M_1(\Gamma_1)}{16} - \frac{q_1}{8}$

Likewise, the contribution of the edges in E_2 to the harmonic index of $\Gamma_1 \vee_Q \Gamma_2$ is given by

$H(\Gamma_1 \vee_Q \Gamma_2) = \sum_{rs\in I(\Gamma_1)} \sum \frac{2}{Dg_{\Gamma_1\vee_Q\Gamma_2}(r)+Dg_{\Gamma_1\vee_Q\Gamma_2}(s)}$

$=\sum_{r\in V(\Gamma_1)} \sum_{s\in I(\Gamma_1)} \frac{2}{Dg_{\Gamma_1}(r)+Dg_{I(\Gamma_1)}(s)}$

$=\sum_{rs\in E(\Gamma_1)} \sum \frac{2}{Dg_{\Gamma_1}(r)+p_2+Dg_{\Gamma_1}(s)+p_2+Dg_{\Gamma_1}(r)+Dg_{\Gamma_1}(s)}$,by lemma 5.1

$$\le \frac{H(\Gamma_1)}{8} + \frac{q_1}{4p_2}$$

contribution of the edges in E_3 to the harmonic index of $\Gamma_1 \vee_Q \Gamma_2$ is given by

$H(\Gamma_1 \vee_Q \Gamma_2) = \sum_{r\in E(\Gamma_2)} \sum_{s\in E(\Gamma_2)} \frac{2}{Dg_{\Gamma_1\vee_Q\Gamma_2}(r)+Dg_{\Gamma_1\vee_Q\Gamma_2}(s)}$

$=\sum_{r\in E(\Gamma_2)} \sum_{s\in E(\Gamma_2)} \frac{2}{Dg_{(\Gamma_2)}(r)+p_1+Dg_{(\Gamma_2)}(s)+p_1}$ by lemma 5.1

$$\le \frac{1}{4}\sum_{rs\in E(\Gamma_2)} \left(\frac{2}{Dg_{\Gamma_2}(r)+Dg_{\Gamma_2}(s)}+\frac{1}{p_1}\right),$$

$$= \frac{H(\Gamma_2)}{4} + \frac{q_2}{4p_1}.$$

Also, the contribution of the edges in E_4 to the harmonic index of $\Gamma_1 \vee_Q \Gamma_2$ is given by

$H(\Gamma_1 \vee_Q \Gamma_2) = \sum_{r\in V(\Gamma_1)} \sum_{s\in V(\Gamma_2)} \frac{2}{Dg_{\Gamma_1\vee_Q\Gamma_2}(r)+Dg_{\Gamma_1\vee_Q\Gamma_2}(s)}$

$=\sum_{r\in V(\Gamma_1)} \sum_{s\in V(\Gamma_2)} \frac{2}{Dg_{\Gamma_1}(r)+p_2+Dg_{\Gamma_2}(s)+p_1}$ by lemma 5.1

$$\le \frac{p_2 ID(\Gamma_1)}{8} + \frac{p_1 ID(\Gamma_2)}{8} + \frac{p_1p_2}{2(p_1+p_2)}.$$

The desired expression for the harmonic index of $\Gamma_1 \vee_Q \Gamma_2$ is obtained by summing the above four expression.

6.Result of theorem VT-join graph:

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The VR_join graph of $\Gamma_1 and \Gamma_2$, denoted by $\Gamma_1 \vee_T \Gamma_2$, is a graph obtained from $T(\Gamma_1)$ and $\Gamma_2$by joining each vertex of $V(\Gamma_1)$ to all the vertex of Γ_2. By using the structure of the graph $\Gamma_1 \vee_T \Gamma_2$, we can easily get the proof of the following lemma

Lemma 6.1: For the connected graph $\Gamma_k, k = 1,2$, the degree of a vertex s in $\Gamma_1 \vee_T \Gamma_2$ is

$$Dg_{\Gamma_1 \vee_T \Gamma_2}(s) = \begin{cases} 2Dg_{\Gamma_1}(s) + |V(\Gamma_2)|, if s \in V(\Gamma_1) \\ Dg_{\Gamma_2}(s) + |V(\Gamma_1)|, if s \in V(\Gamma_2) \\ Dg_{\Gamma_1}(j) + Dg_{\Gamma_1}(t), if e = jt \in I(\Gamma_1) \end{cases}$$

Theorem 6.2: Let Γ_k be a graph with p_k vertices and q_k edges, k=1,2.

Then $H(\Gamma_1 \vee_T \Gamma_2) \leq \frac{H(L(\Gamma_1))}{4} + \frac{M_1(\Gamma_1)}{16} + \frac{5H(\Gamma_1)}{24} + \frac{H(\Gamma_2)}{4} + \frac{p_2 ID(\Gamma_1)}{16} + \frac{p_1 p_2}{2(p_1+p_2)} - \frac{q_1}{8} + \frac{p_1 ID(\Gamma_2)}{8} + \frac{3q_1}{4p_2} + \frac{q_2}{4q_1}$

Proof: we can splitted up $E(\Gamma_1 \vee_R \Gamma_2)$ into two four subsets, namely,

$E_1 = \{rs \in E(\Gamma_1 \vee_T \Gamma_2)/r, s \in I(\Gamma_1)\}$,

$E_2 = \{rs \in E(\Gamma_1 \vee_T \Gamma_2)/r \in V(\Gamma_1), s \in I(\Gamma_1)\}$

$E_3 = \{rs \in E(\Gamma_1 \vee_T \Gamma_2)/rs \in E(\Gamma_2)$

$E_4 = \{rs \in E(\Gamma_1 \vee_T \Gamma_2)/r \in V(\Gamma_1), s \in V(\Gamma_2)\}$and

$E_5 = \{rs \in E(\Gamma_1 \vee_T \Gamma_2)/rs \in E(\Gamma_1)$

Contribution of the edges in E_1 to the harmonic index of $\Gamma_1 \vee_T \Gamma_2$ is given by

$$H(\Gamma_1 \vee_T \Gamma_2) = \sum_{r,s \in I(\Gamma_1)} \frac{2}{Dg_{\Gamma_1 \vee_T \Gamma_2}(r) + Dg_{\Gamma_1 \vee_T \Gamma_2}(s)}.$$

$$= \sum_{rs \in I(\Gamma_1)} \frac{2}{Dg_{I(\Gamma_1)}(r) + Dg_{I(\Gamma_1)}(s)},$$

$$\leq \frac{H(L(\Gamma_1))}{4} + \frac{M_1(\Gamma_1)}{16} - \frac{q_1}{8}$$

In this way, the contribution of the edges in E_2 to the harmonic index of $\Gamma_1 \vee_T \Gamma_2$is given by

$$H(\Gamma_1 V_T \Gamma_2) = \sum_{r \in V(\Gamma_1)} \sum_{s \in I(\Gamma_1)} \frac{2}{Dg_{\Gamma_1 \vee_T \Gamma_2}(r) + Dg_{\Gamma_1 \vee_T \Gamma_2}(s))}$$

$$= \sum_{r \in V(\Gamma_1)} \sum_{s \in I(\Gamma_1)} \frac{2}{Dg_{\Gamma_1}(r) + Dg_{I(\Gamma_1)}(s)}$$

$$\leq \frac{H(\Gamma_1)}{12} + \frac{q_1}{4p_2}$$

Likewise, the contribution of the edges in E_3 to the harmonic index of $\Gamma_1 \vee_T \Gamma_2$ is given by

$$H(\Gamma_1 \vee_T \Gamma_2) = \sum_{r \in E(\Gamma_2)} \sum_{s \in E(\Gamma_2)} \frac{2}{Dg_{\Gamma_1 \vee_T \Gamma_2}(r) + Dg_{\Gamma_1 \vee_Q \Gamma_2}(s)}$$

$$= \sum_{r \in E(\Gamma_2)} \sum_{s \in E(\Gamma_2)} \frac{2}{Dg_{(\Gamma_2)}(r) + p_1 + Dg_{(\Gamma_2)}(s) + p_1}$$

$$\leq \frac{H(\Gamma_2)}{4} + \frac{q_2}{4p_2}.$$

Also, the contribution of the edges in E_4 to the harmonic index of $\Gamma_1 \vee_T \Gamma_2$ is given by

$$H(\Gamma_1 \vee_T \Gamma_2) = \sum_{r \in V(\Gamma_1)} \sum_{s \in V(\Gamma_2)} \frac{2}{Dg_{\Gamma_1 \vee_T \Gamma_2}(r) + Dg_{\Gamma_1 \vee_T \Gamma_2}(s)}$$

$$= \sum_{r \in V(\Gamma_1)} \sum_{s \in V(\Gamma_2)} \frac{2}{2Dg_{\Gamma_1}(r) + p_2 + Dg_{\Gamma_2}(s) + p_1}$$

$$\leq \frac{p_2 ID(\Gamma_1)}{16} + \frac{p_1 ID(\Gamma_2)}{8} + \frac{p_1 p_2}{2(p_1 + p_2)}.$$

The contribution of the edges in E_5 to the harmonic index of $\Gamma_1 \vee_T \Gamma_2$ is given by

$$H(\Gamma_1 \vee_T \Gamma_2) = \sum_{r \in E(\Gamma_1)} \sum_{s \in E(\Gamma_1)} \frac{2}{Dg_{\Gamma_1 \vee_T \Gamma_2}(r) + Dg_{\Gamma_1 \vee_T \Gamma_2}(s)}$$

$$= \sum_{r \in E(\Gamma_1)} \sum_{s \in E(\Gamma_2)} \frac{2}{2Dg_{\Gamma_1}(r) + p_2 + 2Dg_{\Gamma_2}(s) + p_2}$$

$$\leq \frac{H(\Gamma_1)}{8} + \frac{m_1}{2p_2}.$$

The desired expression for the harmonic index of $\Gamma_1 \vee_Q \Gamma_2$ is obtained by summing the above five expression.

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