On Harmonic Index of new Operation of Graphs

S. Uma, N. Velmurugan

ABSTRACT--- The harmonic index of a graph \( G \), denoted by \( H(\Gamma) \), is defined as \( H(\Gamma) = \sum_{r \in E(\Gamma)} \frac{1}{Dg\Gamma(r)} \) where \( Dg\Gamma(r) \) is the degree of a vertex \( r \) in \( G \). In this paper, we concentrate the upper bounds of the harmonic index for four types of derived graphs.

Keywords: Harmonic index, join, subdivision, total graph.
MSC: 05C12, 05C76.

I. INTRODUCTION

A topological index of a simple connected graph is a numerical quantity related to the graph and its does not depend on the labeling or pictorial representation of the graph. It has been applied for studying quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR) extensively in which the biological activity or other properties of molecules are correlated with their chemical structures, see [6]. In this present literature, large number of graph based structure indices have been put forward, that all depend only on the degree of the vertices of the graph. For more details on degree based topological index and on their comparative study can be found in [7, 8, 9, 10, 4, 5].

In 1986, Gutman et al. proposed the first degree based index called Zagreb index which is

\[ M_1(\Gamma) = \sum_{r \in E(\Gamma)} M_2(\Gamma) = \sum_{r \in E(\Gamma)} \frac{1}{Dg\Gamma(r)} \]

Deng et al. [18] studied the relation between the harmonic index of a connected graph \( \Gamma \) and its chromatic number \( \chi(\Gamma) \).

Zhong [11, 12, 13] obtained the maximum and minimum values of the harmonic index for several graphs such as tree, unicyclic graph and graph with girth at least \( k(k \geq 3) \). Further they characterized the harmonic index for corresponding extremal graphs. The comparison between the harmonic index of a connected graph \( \Gamma \) and its matching numbers are established by Lv et al. [16,17]. Shwetha et al. [15] obtained the formulae for the harmonic index of some derived graphs. Since graph operation plays an important role to study the infinity graphs which are derived from the smaller graphs, in this connection, we present several upper bounds for the harmonic index of the \( V \) \( J \) \( \cup \) \( Q \) \( \vee \) join derived graphs.

II. PRELIMINARIES

For a simple connected graph \( \Gamma \), we obtain the following derived graphs.

i. The subgraph formed by joining each vertex of \( \Gamma \) to a new vertex \( y \) is a graph with vertex set \( V(\Gamma) \cup V(y) \) and every vertex of \( \Gamma \) is joined to all the vertices of \( y \).

ii. The total graph formed by adding a new vertex \( y \) to \( \Gamma \) as well as to all the vertices of \( \Gamma \).

iii. The join graph formed by joining each vertex of \( \Gamma \) and \( y \) as well as to all the vertices of \( \Gamma \).

iv. The subdivision graph obtained from \( \Gamma \) by inserting a new vertex into each edge of \( \Gamma \).

The join of two simple connected graphs \( \Gamma_1 \) and \( \Gamma_2 \) denoted by \( \Gamma_1 \cup \Gamma_2 \) is a graph with vertex set \( V(\Gamma_1) \cup V(\Gamma_2) \) and every vertex of \( \Gamma_1 \) is joined to all the vertices of \( \Gamma_2 \).

Eliasi and Taeri [1] proposed the four new operations of the graph based on the Cartesian product. Sarala et al. [2] studied the new operations of the graphs based on the composition. In this sequence, Sarkar et al. [3] discussed another type of graph operation based on join as follows.

Let \( \Gamma = (S, R, Q, T) \) and \( H(\Gamma) \) denotes the set of vertices \( J(\Gamma) \) which are inserted into each edge \( f(\Gamma) \), so that \( V(J(\Gamma)) = V(\Gamma) \cup J(\Gamma) \). The graph \( J(\Gamma) \) is a graph with vertex set \( V(\Gamma) \cup J(\Gamma) \) and edge set \( E(\Gamma) \cup E(J(\Gamma)) \cup \{rs|r \in V(\Gamma), s \in V(J(\Gamma))\} \) obtained from \( J(\Gamma) \).

Lemma 2.1: for two non negative real numbers \( x \) and \( y \) then \( \frac{x}{x+y} \leq \frac{1}{x} + \frac{1}{y} \).

\( V(\Gamma_1) \cup J(\Gamma_2) \) is a graph obtained from the subdivision graph \( S(\Gamma_1) \) and \( J(\Gamma_2) \) by joining.
each vertex of $V(\Gamma_1)$ to all vertex of $(\Gamma_2)$. By using the structure of the graph $\Gamma_1 V_s \Gamma_2$, we can easily get the proof of the following lemma.

Lemma 3.1

For the connected graph $\Gamma_k, k = 1, 2$, the degree of a vertex $r$ in $\Gamma_1 V_s \Gamma_2$ is

$$D_g \Gamma_1 V_s \Gamma_2 (r) = \begin{cases} \frac{1}{2} \left( |\Gamma_1| + |\Gamma_2| \right), & \text{if } r \in \Gamma_1 \\ \frac{1}{2} \left( |\Gamma_1| + |\Gamma_2| \right), & \text{if } r \in \Gamma_2 \\ 2, & \text{if } r \in V(\Gamma_1) \end{cases}$$

Now we find the upper bound for harmonic index of the graphs $\Gamma_1 V_s \Gamma_2$

Theorem 3.2

For the simple connected graph $\Gamma_k$ with $p_k$ vertices and $q_k$ edges $k=1,2$, $H(\Gamma_1 V_s \Gamma_2) \leq \frac{H(\Gamma_1)}{2} + \frac{H(\Gamma_2)}{2} + \frac{p_1 + p_2}{2}$

Proof: By using the definition of harmonic index, we get $H(\Gamma_1 V_s \Gamma_2) = \sum_{r \in \Gamma_1} \sum_{s \in \Gamma_2} \frac{1}{2} \left( |\Gamma_1| + |\Gamma_2| \right)$.

We can split it up $E(\Gamma_1 V_s \Gamma_2)$ into three subsets, namely,

$E_1 = \{ r \in E(\Gamma_1 V_s \Gamma_2) / r \in E(S(\Gamma_1)) \}$,

$E_2 = \{ r \in E(\Gamma_1 V_s \Gamma_2) / r \in E(\Gamma_2) \}$ and

$E_3 = \{ r \in E(\Gamma_1 V_s \Gamma_2) / r \in V(\Gamma_1), s \in V(\Gamma_2) \}$.

Contribution of the edges in $E_1$ to the harmonic index of $\Gamma_1 V_s \Gamma_2$ is given by

$H(\Gamma_1 V_s \Gamma_2) = \sum_{r \in E_1} \sum_{s \in \Gamma_2} \frac{1}{2} \left( |\Gamma_1| + |\Gamma_2| \right)$,

and with equality if and only if $D_g \Gamma_1 (s) = p_2 + 2$.

$p_1 + q_1 = \frac{1}{p_2 + 2} \tag{3.1}$

Likewise, the contribution of the edges in $E_2$ to the harmonic index of $\Gamma_1 V_s \Gamma_2$ is given by

$H(\Gamma_1 V_s \Gamma_2) = \sum_{r \in E_2} \sum_{s \in \Gamma_2} \frac{1}{2} \left( |\Gamma_1| + |\Gamma_2| \right)$,

and with equality if and only if $D_g \Gamma_2 (r) = p_1 + 2$.

$p_1 + q_1 = \frac{1}{p_2 + 2} \tag{3.1}$

Also for the edges in $E_3$, contribution to the harmonic index of $\Gamma_1 V_s \Gamma_2$ is given by

$H(\Gamma_1 V_s \Gamma_2) = \sum_{r \in \Gamma_1} \sum_{s \in \Gamma_2} \frac{1}{2} \left( |\Gamma_1| + |\Gamma_2| \right)$,

and with equality if and only if $D_g \Gamma_1 (s) = p_1 + 2$.

$p_1 + q_1 = \frac{1}{p_2 + 2} \tag{3.1}$

The desired expression for the harmonic index of $\Gamma_1 V_s \Gamma_2$ is obtained by summing the above three expressions.

4. Result of theorem in VR-join graph:

The VR-join graph of $\Gamma_1$ and $\Gamma_2$, denoted by $\Gamma_1 V_\mathcal{R} \Gamma_2$, is a graph obtained from $R(\Gamma_1)$ and $R(\Gamma_2)$ by joining each vertex of $V(\Gamma_1)$ to all the vertices of $V(\Gamma_2)$. By using the structure of the graph $\Gamma_1 V_\mathcal{R} \Gamma_2$, we can easily get the proof of the following lemma.

Lemma 4.1: For the connected graph $\Gamma_i, i = 1, 2$, the degree of a vertex $r$ in $\Gamma_i V_\mathcal{R} \Gamma_i$ is

$D_g \Gamma_i V_\mathcal{R} \Gamma_i (r) = \frac{1}{2} \left( |\Gamma_i| + |\Gamma_i| + 1 \right)$,

and with equality if and only if $D_g \Gamma_i (s) = p_2 + 2$.

$p_1 + q_1 = \frac{1}{p_2 + 2} \tag{3.1}$

The contribution of the edges in $E_1$ to the harmonic index of $\Gamma_1 V_\mathcal{R} \Gamma_2$ is given by

$H(\Gamma_1 V_\mathcal{R} \Gamma_2) = \sum_{r \in \Gamma_1} \sum_{s \in \Gamma_2} \frac{1}{2} \left( |\Gamma_1| + |\Gamma_1| + 1 \right)$,

and with equality if and only if $D_g \Gamma_2 (r) = p_1 + 2$.

$p_1 + q_1 = \frac{1}{p_2 + 2} \tag{3.1}$

The contribution of the edges in $E_2$ to the harmonic index of $\Gamma_1 V_\mathcal{R} \Gamma_2$ is given by

$H(\Gamma_1 V_\mathcal{R} \Gamma_2) = \sum_{r \in \Gamma_1} \sum_{s \in \Gamma_2} \frac{1}{2} \left( |\Gamma_1| + |\Gamma_1| + 1 \right)$,

and with equality if and only if $D_g \Gamma_1 (s) = p_1 + 2$.
The contribution of the edges in $E_3$ to the harmonic index of $\Gamma_1 V_0 \Gamma_2$ is given by
\[
H(\Gamma_1 V_0 \Gamma_2) = \Sigma_{e \in E(\Gamma_1)} 2\sum_{x \in V(\Gamma_1)} \frac{1}{4Dg_1(r) + 3Dg_1(t)} + \frac{1}{4Dg_1(t) + 3Dg_1(s)}
\]

By summing the above four expressions, the desired expression for the harmonic index of $\Gamma_1 V_0 \Gamma_2$ is obtained.

5. Result of theorem VQ-join graph

The VQ-join graph of $\Gamma_1$ and $\Gamma_2$, is graph denoted from the graph by $Q(\Gamma_1)$ and $\Gamma_2$ by joining each vertex of $V(\Gamma_1)$ to every vertex of $\Gamma_2$ and is denoted by $\Gamma_1 V_0 \Gamma_2$. The line graph of $\Gamma$, denoted by $L(\Gamma)$ is the graph whose vertices correspond to the edges of $\Gamma$ with two vertices being adjacent if and only if the corresponding edges in $\Gamma$ have a vertex in common. By using the structure of the graph $\Gamma_1 V_0 \Gamma_2$, we can easily get the proof of the following lemma.

**Lemma 5.1:** For the connected graph $\Gamma$, $i = 0, 1, 2$, the degree of a vertex $r$ in $\Gamma_1 V_0 \Gamma_2$ is
\[
Dg_{\Gamma_1 V_0 \Gamma_2}(r) = \left\{ \begin{array}{ll} Dg_{\Gamma_1}(r) + \frac{1}{2} & \text{if } r \in V(\Gamma_1) \\ Dg_{\Gamma_2}(r) + \frac{1}{2} & \text{if } r \in V(\Gamma_2) \\ Dg_{r}(j) + Dg_{r}(k) & \text{if } e = jk \in I(\Gamma_1) \end{array} \right.
\]

Now we compute the upper bound for harmonic index of the graph $\Gamma_1 V_0 \Gamma_2$.

**Theorem 5.2:** let $\Gamma_k$ be a graph with $p_k$ vertices and $q_k$ edges, $k = 1, 2$. Then
\[
H(\Gamma_1 V_0 \Gamma_2) \leq H(\Gamma_1) + H(\Gamma_2) + \frac{1}{4} p_1 D(\Gamma_1) + \frac{1}{4} p_2 D(\Gamma_2) + \frac{1}{8} \left( \frac{p_1 q_1 + p_2 q_2}{p_1 + p_2} \right) \frac{1}{8}
\]

Proof: By using the definition of harmonic index, we have
\[
H(\Gamma_1 V_0 \Gamma_2) = \Sigma_{r \in E(\Gamma_1 V_0 \Gamma_2)} \frac{2}{4Dg_{\Gamma_1 V_0 \Gamma_2}(r) + 3Dg_{\Gamma_1 V_0 \Gamma_2}(s)}
\]

we can splitted up $E(\Gamma_1 V_0 \Gamma_2)$ into two four subsets, namely,
\[
E_1 = \{ rs \in E(\Gamma_1 V_0 \Gamma_2) / r, s \in E(\Gamma_1) \}, \\
E_2 = \{ rs \in E(\Gamma_1 V_0 \Gamma_2) / r, s \in E(\Gamma_2) \}, \\
E_3 = \{ rs \in E(\Gamma_1 V_0 \Gamma_2) / rs \in E(\Gamma_2) \} and \\
E_4 = \{ rs \in E(\Gamma_1 V_0 \Gamma_2) / r \in V(\Gamma_1), s \in V(\Gamma_2) \}.
\]

6. Result of theorem VT-join graph:
The VR_join graph of $\Gamma_1$ and $\Gamma_2$, denoted by $\Gamma_1 \cup \Gamma_2$, is a graph obtained from $\Gamma(\Gamma_1)$ and $\Gamma(\Gamma_2)$ by joining each vertex of $\Gamma(\Gamma_1)$ to all the vertex of $\Gamma_2$. By using the structure of the graph $\Gamma_1 \cup \Gamma_2$, we can easily get the proof of the following lemma

**Lemma 6.1:** For the connected graph $\Gamma_k$, $k = 1, 2$, the degree of a vertex $s$ in $\Gamma_k \cup \Gamma_j$ is

$$Dg_{\Gamma_k \cup \Gamma_j}(s) = \left\{ \begin{array}{ll}
2Dg_{\Gamma_1}(s) + |V(\Gamma_2)|, & \text{if } s \in V(\Gamma_1) \\
Dg_{\Gamma_1}(s) + |V(\Gamma_2)|, & \text{if } s \in V(\Gamma_2) \\
Dg_{\Gamma_1}(j) + Dg_{\Gamma_1}(t), & \text{if } e = jt \in E(\Gamma_1)
\end{array} \right.$$ 

**Theorem 6.2:** Let $\Gamma_k$ be a graph with $p_k$ vertices and $q_k$ edges, $k = 1, 2$.

Then $H(\Gamma_k)$ satisfies:

$$H(\Gamma_1 \cup \Gamma_2) \leq \frac{H(L(\Gamma_1))}{4} + \frac{M_r(\Gamma_1)}{16} + \frac{5H(\Gamma_1)}{4} + \frac{p_1D(P_1)}{16} + \frac{p_2D(P_2)}{2(p_1 + p_2)} \leq \frac{H(L(\Gamma_1))}{4} + \frac{M_r(\Gamma_1)}{16} + \frac{5H(\Gamma_1)}{4} + \frac{q_1}{8} + \frac{p_1D(P_1)}{2(p_1 + p_2)} + \frac{q_2}{8} + \frac{p_2D(P_2)}{2(p_1 + p_2)}.$$

**Proof:** We can split up $E(\Gamma_1 \cup \Gamma_2)$ into two four subsets, namely:

$E_1 = \{ rs \in E(\Gamma_1 \cup \Gamma_2) / r, s \in E(\Gamma_1) \}$,

$E_2 = \{ rs \in E(\Gamma_1 \cup \Gamma_2) / r \notin V(\Gamma_1), s \in E(\Gamma_1) \}$,

$E_3 = \{ rs \in E(\Gamma_1 \cup \Gamma_2) / r \notin E(\Gamma_1) \}$,

$E_4 = \{ rs \in E(\Gamma_1 \cup \Gamma_2) / r \notin V(\Gamma_2), s \in V(\Gamma_2) \}$,

and

$E_5 = \{ rs \in E(\Gamma_1 \cup \Gamma_2) / r \notin E(\Gamma_1) \}$.

The desired expression for the harmonic index of $\Gamma_1 \cup \Gamma_2$ is obtained by summing the above five expression.

REFERENCES