# On a Generalized Notion of Anti-fuzzy Subgroup and Some Characterizations

**Sudipta Gayen, Sripati Jha, Manoranjan Singh, Ranjan Kumar**

***Abstract: We have presented a new notion of anti-fuzzy subgroup (AFS). For this, we have considered general t-conorm. The main contributions of this paper are fourfold: (1) we have proposed a new notion of anti-fuzzy subgroup (AFS), (2) we have also defined infimum image of a fuzzy set, (3) Furthermore, we have defined subgroup generated anti-fuzzy subgroup (SGAFS), function generated anti-fuzzy subgroup (FGAFS) and (4) we have shown that an AFS proposed earlier belong to a special class of subgroup generated anti-fuzzy subgroup (SGAFS). To justify our proposed notion we have discussed some drawbacks of the existing notion of AFS with numerical examples. Finally, we have concluded that our proposed notion is superior to the existing one.***

***Index Terms: Infimum image; Subgroup generated anti-fuzzy subgroup; Function generated anti-fuzzy subgroup.***

## I.INTRODUCTION

Classical abstract algebra or crisp abstract algebra is one of the essential building blocks in mathematics. It has an immense impact on different applied as well as pure fields. But it is based on classical (crisp) set theory, which is inadequate while solving real-life problems because every object of our surrounding carries some degree of uncertainty. To deal with this kind of uncertainty or fuzziness Zadeh [1] introduced a fuzzy set theory. Since the very first introduction of fuzzy set theory, many researchers have implemented the concept on several real-life applications such as transportation problem [2], shortest path problem [3], ERP database, statistical analysis etc. In fuzzy abstract algebra, some pioneers have introduced the concepts like fuzzified versions of subgroup [4], ideal [4], normal subgroup [5], ring [6], field [7] etc. The concepts of $T$-norm and $T$-conorm were introduced by Schweizer and Sklar [8] which are nothing but the extension of triangle inequalities. The most well-known $T$-operators are $T_1(m,u) = Min\{m,u\}$ and $T_1^*(m,u) = Max\{m,u\}$. These have been widely used in modeling of fuzzy logic controllers [9] and other decision-making techniques [10]. But some scientific studies suggest that there are alternative types of $T$-operators [11] which are superior in some aspects, mainly in decision-making techniques [12]. Rosenfeld [4] introduced the concept of fuzzy subgroup (FS) with respect to $T_1$. Furthermore, Anthony and Sherwood [13] redefined FS using general $t$-norm $T$ suggesting some examples in which Rosenfeld's [4] version of FS was giving contradicting results.

**Manuscript published on 28 February 2019.**
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Also, the concepts of Subgroup generated fuzzy subgroup (SGFS), as well as Function generated fuzzy subgroup (FGFS) were introduced by Anthony and Sherwood [14]. They have shown that SGFS and FGFS are essentially equivalent. Also, a FS corresponding to $T_1$ is subgroup generated and the generating class carries a special subclass of measure.

Biswas [15] introduced the notion of AFS which was based on $T_1^*$. He proved that an AFS is nothing but the complement of a FS with respect to $T_1$. Later on, some mathematicians have introduced some special types of AFS like intuitionistic AFS [16] which was based on [17] [18], $(\lambda,\mu)$ AFS [19] which was based on [20]. Also, the idea of AFS has been implemented to ideal, ring, field [21], [22] etc. Some researchers have modified the idea of AFS and implemented that in BCK/BCI/BF/BH/BE/BCH /CI-Algebras [23], [24], [25], [26], [27] which are widely used in artificial intelligence [28], computer science [29], medical science, control engineering [30], decision theory [31], expert systems, operations research [32], pattern recognition [33], robotics [34] and other fields. As Biswas's [15] version of AFS is based on $t$-conorm $T_1^*$ it has some drawbacks. There are some examples which contradict the notion of AFS which was proposed earlier. But it can be redefined with respect to other $t$-conorms like $T_2^*(m,u) = m+u-mu$ or $T_3^*(m,u) = Min\{m+u,1\}$ and some drawbacks can be omitted. Hence AFS can be generalized using general $t$-conorm $T^*$. Depending on that other concept like intuitionistic AFS, $(\lambda,\mu)$ AFS, anti-fuzzy ideal etc can be generalized. Even, a generalized version of an AFS can be implemented in various algebras like BCK/BCI/BF/BH/BE/BCH/CI etc. which have extensive applications in various fields. Also, some new concepts like SGAFS and FGAFS can be defined. So far, the notion of AFS has been explored by numerous researchers. We believe that none has considered our approach. In this paper the main contributions are:

- We have redefined AFS on the basis of general t-conorm $T^*$.
- We have also discussed some drawbacks of the existing notion of AFS with some numerical examples. Additionally, we have reasoned that our proposed notion is superior.
- Moreover, we have defined the infimum image of a fuzzy set and proposed a new theory.
- Furthermore, we have defined SGAFS and FGAFS. We have also proved that they are equivalent to each other and shown that AFS proposed by Biswas [15] belong to a family of SGAFS which is simply ordered in terms of set inclusion.


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This paper has been organized as following: Section II includes definitions of FS, AFS, $T$-norm, $T$-conorm, and discussion of some drawbacks of the existing notion of AFS [15]. Section III contains our proposed notions of AFS with respect to some $t$-conorms and the general $t$-conorm, infimum image of a fuzzy set with some important propositions and some properties of AFS. In Section IV we have mentioned our proposed notions of SGAFS, FGAFS with some theorems. Finally, we conclude that our proposed notion of AFS is superior to any existing one.

## II. PRELIMINARIES

**Definition 2.1.** [4] A fuzzy subset $\delta$ of a group $H$ is termed as a FS of $H$ if $\forall\ m,\ u \in H$ the subsequent conditions are fulfilled:

(i) $\delta(mu) \geq Min\{\delta(m), \delta(u)\}$

(ii) $\delta(m^{-1}) \geq \delta(m)$.

**Definition 2.2.** [15] A fuzzy subset $\gamma$ of a group $H$ is termed as an AFS of $H$ if $\forall\ m,\ u \in H$, subsequent conditions are fulfilled:

(i) $\gamma(mu) \leq Max\{\gamma(m), \gamma(u)\}$,

(ii) $\gamma(m^{-1}) \leq \gamma(m)$.

### A. T-norm and T-conorm

**Definition 2.3.** [11] A function $T: [0,1] \times [0,1] \to [0,1]$ is termed as a $t$-norm iff $\forall\ m,\ u,\ t \in [0,1]$, subsequent conditions are fulfilled:

(i) $T(m, 1) = m$,

(ii) $T(m,u) = T(u,m)$,

(iii) $T(m,u) \leq T(t,u)$ $if\ m \leq t$,

(iv) $T(m, T(u,t)) = T(T(m,u),t)$.

Some examples of $t$-norms that we frequently use are $Min\{m,u\}$, $Max\{m+u-1,0\}$ and $Prod(m,u) = mu$. Let's rename these usual $t$-norms as $T_1(m,u) = Min\{m,u\}$, $T_2(m,u) = Prod(m,u)$ and $T_3(m,u) = Max\{m+u-1, 0\}$.

**Definition 2.4.** [11] A function $T^*: [0,1] \times [0,1] \to [0,1]$ is called a t-conorm iff $\forall\ m,\ u,\ t \in [0,1]$, subsequent conditions are fulfilled:

(i) $T^*(m, 0) = m$,

(ii) $T^*(m,u) = T^*(u,m)$,

(iii) $T^*(m,u) \leq T^*(t,u)$ $if\ m \leq t$,

(iv) $T^*(m, T^*(u,t)) = T^*(T^*(m,u),t)$.

Some examples of t-conorms that we frequently encounter are $Max\{m,u\}$, $Min\{m+u, 1\}$ and $m+u-mu$. Let's rename these usual t-conorms as $T_1^*(m,u) = Max\{m,u\}$, $T_2^*(m,u) = m+u-mu$ and $T_3^*(m,u) = Min\{m+u, 1\}$.

### B. The redefined notion of FS

In Definition 2.1 of FS instead of using $T_1$ if we use $T_2$ we can get a generalized notion of FS. Even further instead of using $T_2$ if we use $T_3$ we will get morea generalized version of FS. So, in general we can redefine FS as

**Definition 2.5.** [13] A fuzzy subset $\delta$ of a group $H$ is termed as a FS of $H$ with respect to $T$ if $\forall\ m,\ u \in H$, subsequent conditions are fulfilled:

(i) $\delta(mu) \geq T(\delta(m), \delta(u))$

(ii) $\delta(m^{-1}) = \delta(m)$.

### C. A list of abbreviations used throughout this paper

**FS** stands for "fuzzy subgroup".
**AFS** stands for "anti-fuzzy subgroup".
**SGAFS** stands for "subgroup generated anti-fuzzy subgr-oup".
**FGAFS** stands for "function generated anti-fuzzy subgro-up".
$\boldsymbol{T}$ stands for "general t-norm".
$\boldsymbol{T^*}$ stands for "general t-conorm".
$\boldsymbol{T_1^*}$ stands for "$t$-conorm $T_1^*(m,u) = Max\{m,u\}$".
$\boldsymbol{T_2^*}$ stands for "$t$-conorm $T_2^*(m,u) = m+u-mu$".
$\boldsymbol{T_3^*}$ stands for "$t$-conorm $T_3^*(m,u) = Min\{m+u, 1\}$".
$\boldsymbol{K \lesssim H}$ stands for "$K$ is a subgroup of $H$".

### D. Limitation and drawback of the existing notion of AFS

A possible way to generate FS would be assuming the value of a FS at a point, say $m$ be the probability such that $m$ can be found in an arbitrarily seclected subgroup. Again, the probability that $m$ is not in an arbitrarily seclected subgroup can be denoted as the value of an AFS at that point.

**Example 2.1.** [13] Let $(\mathbb{Z}, +)$ be a group with respect to the usual addition. Let $S_i = \{k \in \mathbb{Z}: i|k\}$. Let $\Delta = \{S_2,\ S_3,\ S_5\}$, $\mathcal{V}$ be the power set of $\Delta$ and $\eta: \mathcal{V} \to \mathbb{R}$ be a probability measure with $\eta(S_2) = \eta(S_3) = \eta(S_5) = \frac{1}{3}$. Now $\gamma_\Delta(6) = \frac{1}{3}$, $\gamma_\Delta(15) = \frac{1}{3}$ and $\gamma_\Delta(21) = \frac{2}{3}$. Notice that $\gamma_\Delta(21) > Max\{\gamma_\Delta(6), \gamma_\Delta(15)\}$. So, $\gamma_\Delta$ can not be an AFS of $(\mathbb{Z}, +)$ according to definition 2.2.

**Example 2.2.** [13] Let $(\Delta, \mathcal{V}, \eta)$ is forming a probability space where $\Delta$ represents $[0,1]$, $\mathcal{V}$ represents a collection of Borel subsets of $\Delta$ and $\eta$ is corresponding Lebesgue measure. Again, let $\mathcal{R}$ represents a set consisting of all the random variables on $(\Delta, \mathcal{V}, \eta)$ and $\mathbb{Z}$ represents the set of integers. Let $\gamma_\mathbb{Z}: \mathbb{R} \to [0,1]$ is defined as $\gamma_\mathbb{Z}(P_{rv}) = 1 - \eta(\{t \in \Delta: P_{rv}(t) \in \mathbb{Z}\})$ or in other words $\gamma_\mathbb{Z}(P_{rv})$ represents the probability that $P_{rv}$ is not in $\mathbb{Z}$. Let us define two random variables $P_{rv}$ and $Q_{rv}$ by

$$P_{rv}(t) = 1\ \ if\ \ t \in [0, \tfrac{1}{2}]$$
$$= \tfrac{1}{2}\ \ if\ \ t \in (\tfrac{1}{2}, 1]$$

and

$$Q_{rv}(t) = \tfrac{1}{2}\ \ if\ \ t \in [0, \tfrac{1}{3}]$$
$$= 1\ \ if\ \ t \in (\tfrac{1}{3}, 1]$$

Wherefrom, $(P_{rv} + Q_{rv})(t) = \frac{3}{2}$ $if\ \ t \in [0, \frac{1}{3}]$
$= 2$ $if\ \ t \in (\frac{1}{3}, \frac{1}{2}]$


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$= \frac{3}{2}$ $if$ $t \in (\frac{1}{2}, 1]$

Then $\gamma_{\mathbb{Z}}(P_{rv}) = \frac{1}{2}$, $\gamma_{\mathbb{Z}}(Q_{rv}) = \frac{1}{3}$ and $\gamma_{\mathbb{Z}}(P_{rv} + Q_{rv}) = \frac{5}{6}$.

Again, $\gamma_{\mathbb{Z}}(P_{rv} + Q_{rv}) = \frac{5}{6} > Max\{\frac{1}{2}, \frac{1}{3}\} = Max\{P_{rv}, Q_{rv}\}$ . Hence $\gamma_{\mathbb{Z}}$ can not be an AFS of $\mathcal{R}$ according to definiti- on 2.2. In both the above-mentioned examples we have chosen membership grade of a point corresponding to $\gamma$ as the probability of not finding that point in a randomly seclected subgroup. Cosequently, our choices of $\gamma$ beca- me legitimate candidates for becoming AFSs. But in both the cases we have found contradictions. These examples give us enough reasons to propose a new notion of AFS.

## III.A PROPOSED NOTION OF AFS

In this section using different $t$-conorms, we have generalized AFS.

**Definition 3.1.** A fuzzy subset $\gamma$ of a group $H$ is termed as an AFS of $H$ if $\forall$ $m, u \in H$, subsequent conditions are fulfilled:

(i) $\gamma(mu) \leq \gamma(m) + \gamma(u) - \gamma(m)\gamma(u)$
(ii) $\gamma(m^{-1}) \leq \gamma(m)$
.

**Definition 3.1.** A fuzzy subset $\gamma$ of a group $H$ is termed as an AFS of $H$ if $\forall$ $m, u \in H$, subsequent conditions are fulfilled:

(i) $\gamma(mu) \leq Min\{\gamma(m) + \gamma(u), 1\}$,
(ii) $\gamma(m^{-1}) \leq \gamma(m)$.

In Definition 3.1 and Definition 3.2 we have used t-conorms respectively as $T_2^*$ and $T_3^*$ which in turn have generalized Definition 2.2. Hence in general we can redefine AFS as

**Definition 3.3.** A fuzzy subset $\gamma$ of a group $H$ is termed as an AFS of $H$ with respect to $T^*$ if $\forall$ $m, u \in H$, the subsequent conditions are fulfilled:

(i) $\gamma(mu) \leq T^*(\gamma(m), \gamma(u))$
(ii) $\gamma(m^{-1}) \leq \gamma(m)$.

Notice that $\gamma_\Delta$ defined in Example 2.1 and $\gamma_{\mathbb{Z}}$ defined in Example 2.2 can not be AFS even if our choice of $t$-conorm is $T_2^*$. But, if our choice of $t$-conorm is $T_3^*$, then they will be AFS.

**Proposition 3.1.** Let $H$ be a group and $\varDelta$ be a set consisting of all the subgroups of $H$ which is not simply ordered with respect to set inclusion. Let $\mathcal{V}$ be a $\sigma$-algebra of some subsets of $\varDelta$. Let $K_m = \{K \in \varDelta : m \in K\}$ with $K_m \in \mathcal{V}$ for all $m \in G$. Let $(\varDelta, \mathcal{V}, \eta)$ is forming a probability space where m is a probability measure on $(\varDelta, \mathcal{V})$. Let us define a funtion $\gamma_\varDelta : H \to [0,1]$ such that $\gamma_\varDelta(m) = 1 - \eta(K_m)$ for all $m \in H$. The function $\gamma_\varDelta$ will form an AFS of $H$ with respect to the $T_3^*$.

**Proof.** Let $K \in K_m \cap K_u$ and $m, u \in K$.
As $K \lesssim H$, $mu \in K$ and hence $K \in K_{mu}$.
Also, $K_m \cap K_u \subset K_{mu}$.
Now,

$$\begin{aligned}\gamma_\varDelta(mu) &= 1 - \eta(K_{mu})\\ &\leq 1 - \eta(K_m \cap K_u)\\ &\leq 1 - \eta(K_m) - \eta(K_u) + \eta(K_m \cup K_u)\\ &= 1 - \eta(K_m) + 1 - \eta(K_u)\\ &= \gamma_\varDelta(m) + \gamma_\varDelta(q)\end{aligned}$$

Again, $\gamma_\varDelta(mu) \leq 1$.
Therefore, $\gamma_\varDelta(mu) \leq T_3^*(\gamma_\varDelta(m), \gamma_\varDelta(q))$.
Now, let $K \in K_m$. So, $m \in K$ and $K$ being a subgroup of $H$, $m^{-1} \in K$ and hence $K \in K_{m^{-1}}$.
Therefore, $K_m \subset K_{m^{-1}}$ and $\eta(K_{m^{-1}}) \geq \eta(K_m)$
i.e. $1 - \eta(K_{m^{-1}}) \leq 1 - \eta(K_m)$or,$\gamma_\varDelta(m^{-1}) \leq \gamma_\varDelta(p)$. Hence, $\gamma_\varDelta$ forms an AFS with respect to $T_3^*$.

**Proposition 3.2.** In Proposition 3.1 if our choice of $\varDelta$ is such that it is simply ordered in terms of set inclusion, then $\gamma_\varDelta$ will form an AFS with respect to $T_1^*$ (definition 2.2).

**Proof.** Let $m, u \in H$. As $\varDelta$ is simply ordered in terms of set inclusion, either $K_m \subset K_u$ or $K_u \subset K_m$. Let $K_m \subset K_u$ and $K \subset K_m$. Then $K \subset K_u$ and $u \in K$. As $K \lesssim H$, $mu \in K$ and hence $K \in K_{mu}$. So, $K_m \subset K_{mu}$.
Now,

$$\begin{aligned}\eta(K_{mu}) &\geq \eta(K_m)\\ 1 - \eta(K_{mu}) &\leq 1 - \eta(K_m)\\ \gamma_\varDelta(K_{mu}) &\leq Max\{1 - \eta(K_m), 1 - \eta(K_u)\}\\ \gamma_\varDelta(K_{mu}) &\leq Max\{\gamma_\varDelta(K_m), \gamma_\varDelta(K_u)\}\end{aligned}$$

i. e. $\gamma_\varDelta(K_{mu}) \leq T_1^*(\gamma_\varDelta(K_m), \gamma_\varDelta(K_u))$
Again, following the same logic mentioned in Proposition 3.1, we can prove $\gamma_\varDelta(m^{-1}) \leq \gamma_\varDelta(p)$. So, $\gamma_\varDelta$ forms an AFS with respect to $T_1^*$.

### A. Some properties of the proposed notion of AFS

Most of the properties of AFS with respect to $T_1^*$ proposed in [15] remain valid under other $t$-conorms but with some minor differences. For instance, if $\gamma$ is an anti-fuzzy subgroupoid of $H$ on the basis of $T_1^*$ then all the lower level subsets of $\gamma$ become subgropoids of $H$. This result do not hold for other $t$-conorms.
However, stronger generalized forms of most of the results can be given using general $t$-conorms. In this section, some generalized and new results have been discussed.

**Definition 3.4.** A fuzzy set $\gamma$ in a set $H$ has an infimum if for any $K \subset H$ there exits $k_0 \in K$ such that $\gamma(k_0) = \inf_{k \in K} \gamma(k)$.

**Definition 3.5.** Let $\gamma$ be a fuzzy set in $H$ and $f$ be a function deined on $H$. The fuzzy set $\gamma_f$ defined on $f(H)$ as $\gamma_f(u) = \inf_{m \in f^{-1}(u)} \gamma(m)$ is called the infimum image of $\gamma$ under $f$.

**Proposition 3.3.** Let $\gamma$ be an anti-fuzzy subgroupoid on $H$ on the basis of a continuous $t$-conorm $T^*$ and $f$ be a homomorphism on $H$ then the infimum image of $\gamma$ i.e. $\gamma_f$ is an anti-fuzzy subgroupoid on $f(H)$ with respect to $T^*$.


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**Proof.** Let $u_1,\ u_2 \in f(H)$ and $S_1 = f^{-1}(u_1)$, $S_2 = f^{-1}(u_2)$ and $S_{12} = f^{-1}(u_1 u_2)$.
Let, $S_1 S_2 = \{m \in H: m = m_1 m_2$ for some $m_1 \in S_1$ and $m_2 \in S_2\}$
Let $m \in S_1 S_2$ then $m = m_1 m_2$ for some $m_1 \in S_1$ and $m_2 \in S_2$.
Here $f$ being a homomorphism $f(m) = f(m_1 m_2) = f(m_1) f(m_2) = u_1 u_2$ and hence $S_1 S_2 \subseteq S_{12}$.
Now,

$$\gamma_f(u_1 u_2) = \inf_{m \in S_{12}} \gamma(m) \le \inf_{m \in S_1 S_2} \gamma(m)$$
$$\le \inf_{m_1 \in S_1, m_2 \in S_2} \gamma(m_1 m_2)$$
$$\le \inf_{m_1 \in S_1, m_2 \in S_2} T^*(\gamma(m_1), \gamma(m_2))$$

As $T^*$ is continuous for all $\epsilon > 0$ there exits $\gamma > 0$ so that if $m_1^* \le \inf_{m_1 \in S_1} \gamma(m_1) + \gamma$ and $m_2^* \le \inf_{m_2 \in S_2} \gamma(m_2) + \gamma$ then $T^*(m_1^*, m_2^*) \le T^*(\inf_{m_1 \in S_1} \gamma(m_1), \inf_{m_2 \in S_2} \gamma(m_2)) + \epsilon$ . Let $s_1$ and $s_2 \in S_2$ such that $\gamma(s_1) \le \inf_{m_1 \in S_1} \gamma(m_1) + \gamma$ and $\gamma(s_1) \le \inf_{m_2 \in S_2} \gamma(m_2) + \gamma$.
Then,

$$T^*(\gamma(s_1), \gamma(s_2)) \le T^*(\inf_{m_1 \in S_1} \gamma(m_1), \inf_{m_2 \in S_2} \gamma(m_2)) + \epsilon$$

Now, $\gamma_f(u_1 u_2) \le \inf_{m_1 \in S_1, m_2 \in S_2} T^*(\gamma(m_1), \gamma(m_2))$

$$\le T^*(\inf_{m_1 \in S_1} \gamma(m_1), \inf_{m_2 \in S_2} \gamma(m_2))$$
$$= T^*(\gamma_f(u_1), \gamma_f(u_2))$$

$\therefore\ \gamma_f$ is an anti-fuzzy subgroupoid on $f(H)$ with respect to $T^*$. A relation between t-norm $T$ and corresponding dual t-conorm $T^*$ is $T(m,u) = 1 - T^*(1-m, 1-u)$.

**Proposition 3.4.**

(i) $\delta$ is a FS of $H$ on the basis of $T_1$ if and only if (iff) its complement, $\delta^c$ is an AFS of $H$ with respect to $T_1^*$.

(ii) $\delta$ is a FS of $H$ on the basis of $T_2$ iff its complement, $\delta^c$ is an AFS of $H$ corresponding to $T_2^*$.

(iii) $\delta$ is a FS of $H$ corresponding to $T_3$ iff its complement $\delta^c$ is an AFS of $H$ on the basis of $T_3^*$.

Or, In general, $\delta$ is a FS of $H$ corresponding to $T$ iff its complement, $\delta^c$ is an AFS of $H$ on the basis of $T^*$.

**Proof.** As $\delta$ is a FS of $H$ with respect to $T$,

$$\delta(mu) \ge T(\delta(m), \delta(u))$$
$$1 - \delta^c(mu) \ge T(1 - \delta^c(m), 1 - \delta^c(u))$$
$$1 - \delta^c(mu) \ge 1 - T^*\big(\delta^c(m), \delta^c(u)\big)$$
$$\delta^c(mu) \le T^*(\delta^c(m), \delta^c(u)).$$

Again $\delta(m^{-1}) \ge \delta(m)$ or $1 - \delta^c(m^{-1}) \ge 1 - \delta^c(m)$ i.e. $\delta^c(m^{-1}) \le \delta^c(m)$. Hence $\delta^c$ is an AFS of $H$ with respect to $T^*$.

**Proposition 3.5.** Let $\gamma$ be an AFS of a group H with respect to $T^*$, then $\forall m, u \in H$ $\gamma(mu^{-1}) \le T^*(\gamma(m), \gamma(u))$. Again, if $\gamma$ is a fuzzy subset of H and $T^*$ be a t-conorm with $\forall m, u \in H$ $\gamma(mu^{-1}) \le T^*(\gamma(m), \gamma(u))$ and $\gamma(e) = 0$ (e is the nutral element of H), then $\gamma$ is an AFS of H with respect to $T^*$.

**Proof.** Let $\gamma$ be an AFS of a group $H$ with respect to $T^*$. Then, by definition $\forall m, u \in H$, $\gamma(mu^{-1}) \le T^*(\gamma(m), \gamma(u^{-1})) = T^*(\gamma(m), \gamma(u))$ (as $\gamma(u^{-1}) = \gamma(u)$).
Again, let $\forall m, u \in H$ $\gamma(mu^{-1}) \le T^*(\gamma(m), \gamma(u))$ and $\gamma(e) = 0$ . Then $\gamma(u^{-1}) = \gamma(eu^{-1}) \le T^*(\gamma(e), \gamma(u)) = T^*(0, \gamma(u)) = \gamma(u)$ i.e. $\gamma(u^{-1}) \le \gamma(u)$.
Moreover,

$$\gamma(mu) = \gamma(m(u^{-1})^{-1}) \le T^*(\gamma(m), \gamma(u^{-1}))$$
$$= T^*(\gamma(m), \gamma(u)).$$

Hence $\gamma$ is an AFS of $H$.

**Proposition 3.6.** Let $\gamma$ be an AFS of a group $H$ on the basis of $T^*$. The set $K = \{m \in H: \gamma(m) = 0\}$ is a subgroup of $H$.

**Proof.** Let $m, u \in K$ then

$$\gamma(mu^{-1}) \le T^*\big(\gamma(m), \gamma(u^{-1})\big)$$
$$= T^*\big(\gamma(m), \gamma(u)\big)$$
$$= T^*(0,0) = 0.$$

So, $\gamma(mu^{-1}) = 0$ and hence $mu^{-1} \in K$
i.e. $K$ is a subgroup of $H$.

**Proposition 3.7** Let $\gamma$ be an AFS of a group $H$ with respect to $T^*$ with $\gamma(mu^{-1}) = 0$. Then $\gamma(m) = \gamma(u)$.
**Proof.** $\gamma(m) = \gamma((mu^{-1})u) \le T^*(\gamma(mu^{-1}), \gamma(u)) = T^*(0, \gamma(u)) = \gamma(u)$.

## IV.PROPOSED NOTIONS OF SUBGROUP GENERATED AFS (SGAFS) & FUNCTION GENERATED AFS (FGAFS)

Some AFS can be generated in a particular way. For inst-ance, as mentioned in Subsection II.D, the value of an AFS at a particular point $m$ can be represented as the probabi-lity that $m$ will not be found in an arbitrarily selected subgroup.

**Theorem 4.1.** Let $H$ be a group with $\Delta$ representing a set consisting of all the subgroups of $H$ which is not sim-ply ordered in terms of set inclusion. Let $\mathcal{V}$ represents a σ-algebra consisting of some subsets of $\Delta$. Let $K_m = \{K \in \Delta : m \in K\}$ with $K_m \in \mathcal{V}$ for every $m \in G$ . Also, Let $(\Delta, \mathcal{V}, \eta)$ is forming a probability space. Let us defin-e a funtion $\gamma: H \to [0,1]$ so that $\gamma(m) = 1 - \eta(K_m)$ for every $m \in H$. The function $\gamma$ will form an AFS of $H$ with respect to $T_3^*$.

The proof of Theorem 4.1 has already been given in Proposition 3.1. An AFS obtained by the above-mentioned way is called SGAFS.

Another type of AFS can be generated by considering a point $m$ which moves randomly through a group. One can calculate the probability of not finding that point in a randomly seclected subgroup of that group.

**Theorem 4.2.** Let $(H, +)$ be a group, $K \lesssim H$ and $\Delta$ be a set consisting of all the subgroups of $H$ which is not simply ordered in terms of set inclusion.

Again, Let $\mathcal{V}$ represents a $\sigma$-algebra consisting of some subsets of $\Delta$. Let $(\Delta, \mathcal{V}, \eta)$ is forming a probability space. Also, Let us consider $(F, \oplus)$ as a group consisting of all the functions from $\Delta$ onto $H$ with respect to $\oplus$ defined as the pointwise addition. Again, let us assume that for each $g \in F$, $L_g = \{\omega \in \Delta: g(\omega) \in K\} \in \mathcal{V}$ . Let us define a funtion $\gamma': F \to [0,1]$ such that $\gamma'(g) = 1 - \eta(L_g)$ for all $g \in F$. The function $\gamma'$ will form an AFS of $F$ corresponding to $T_3^*$.

**Proof.** Let $g_1, g_2 \in F$ and $\omega \in L_{g_1} \cap L_{g_2}$. So,


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$g_1(w) \in K$ and $g_2(w) \in K$. $K$ being a subset of $H$, $g_1(w) + g_2(w) = (g_1 \oplus g_2)(w) \in K$ and hence $L_{g_1} \cap L_{g_2} \subseteq L_{g_1 \oplus g_2}$.

Now

$$\gamma'(g_1 \oplus g_2) = 1 - \eta(L_{g_1 \oplus g_2}) \leq 1 - \eta(L_{g_1} \cap L_{g_2}) = 1 - \eta(L_{g_1}) - \eta(L_{g_2}) + \eta(L_{g_1} \cup L_{g_2}) = 1 - \eta(L_{g_1}) + 1 - \eta(L_{g_2}) = \gamma'(g_1) + \gamma'(g_2)$$

Again, $\gamma'(g_1 \oplus g_2) \leq 1$.

So, $\gamma' \leq Min\{\gamma'(g_1) + \gamma'(g_2), 1\} = T_3^*(\gamma'(g_1), \gamma'(g_2))$ .

Also, note that

$$L_{\ominus g_1} = L_{g_1} \text{ and } \gamma'(\ominus g_1) = 1 - \eta(L_{\ominus g_1}) = 1 - \eta(L_{g_1}) = \gamma'(g_1).$$

Hence $\gamma'$ forms an AFS of $F$ with respect to $T_3^*$.

An AFS obtained as above is called function generated AFS or FGAFS.

**Theorem 4.3.** Every FGAFS is SGAFS.

**Proof.** Let $(H,+)$, $K$, $\Delta$, $(F,\oplus)$, $(\Delta, \mathcal{V}, \eta)$, $L_g$ and $\gamma'$ represent the same as mentioned in Theorem 4.2. Let $\Delta'$ represents a collection of all the subgroups of $F$ and, $I_\omega = \{g \in F : g(\omega) \in K\}$ $\forall \omega \in \Delta$. Clearly, $\forall \omega \in \Delta$, $I_\omega$ is a subgroup of $F$. Let $\forall \omega \in \Delta$, $\mu: \Delta \to F$ is defined as $\mu(\omega) = I_\omega$. Let $\mathcal{V}'$ represents a $\sigma$-algebra consisting of some subsets of $\Delta'$ and $(\Delta', \mathcal{V}', \eta')$ be the corresponding probability space which have been induced by $\mu$ and $(\Delta, \mathcal{V}, \eta)$. So, any collection, say $C$ of subgroups of $F$ is measurable iff $\mu^{-1}(C) = \{\omega \in \Delta : I_\omega \in C\} \in \mathcal{V}$ and $\eta'(C) = \eta(\mu^{-1}(C))$ . Let $J_g = \{J \in \Delta' : g \in J\}$. Notice that $\mu^{-1}(J_g) = L_g \in \mathcal{V}$ and hence $J_g \in \mathcal{V}'$ and $\eta'(J_g) = \eta(L_g)$. Now by Theorem 4.1, we can define a function $\gamma: F \to [0,1]$ such that $\gamma(g) = 1 - \eta'(J_g)$ for all $g \in F$ which is a SGAFS of $F$. Again, as $\gamma(g) = 1 - \eta'(J_g) = 1 - \eta(J_g) = \gamma'(g)$ $\forall g \in F$, $\gamma = \gamma'$. So, $\gamma'$ is SGAFS.

**Theorem 4.4.** Every SGAFS is isomorphic to a FGAFS.

**Proof.** Let $\gamma$ be a SGAFS on a group $H$ with respect to multiplication. Let $(\Delta, \mathcal{V}, \eta)$ represent the same notion as mentioned in Theorem 4.3. Again, Let $P_H = \Pi_{i=1}^{n} H$ and $P_K = \Pi_{K \in \Delta} K$, where $n$ is the cardinality of $\Delta$. Clearly, $P_H$ forms a group and $P_K$ is a subgroup of $P_H$ with respect to the component wise addition over multiplication in $H$.

Let $\forall m \in H$, $\phi_m: \mathcal{V} \to P_H$ be such that

$$\phi_m(K) = \psi_{m,K}, \text{ where } \psi_{m,K}(K^*) = \begin{cases} e & if\ K^* \neq K \\ m & if\ K^* = K \end{cases}.$$

Let $F = \{\phi_m : m \in H\}$ and $\oplus: F \to P_H$ be defined as $(\phi_m \oplus \phi_u)(K) = \psi_{m,K} + \psi_{u,K} = \psi_{mu,K} = \phi_{mu}(K)$ $\forall K \in \mathcal{V}$. Consequently, $(F,\oplus)$ is a group. Let $\xi: H \to F$ with $\xi(m) = \phi_m$ $\forall m \in H$. This function $\xi$ is an isomorphism. Let $\forall m \in H$ $K_m$ be the subset of $\Delta$ as mentioned in Theorem 4.1 and $L_{\phi_m} = \{K \in \Delta : \phi_m(K) \in P_K\}$. Clearly, $L_{\phi_m} = K_m \in \mathcal{V}$. Here $(P_H, +)$, $P_K$, $(\Delta, \mathcal{V}, \eta)$ and $(F,\oplus)$ satisfy the same descriptions of $(P_H, +)$, $P_K$, $(\Delta', \mathcal{V}', \eta')$ and $(F,\oplus)$ as mentioned in Theorem 4.2. Hence by Theorem 4.2, there will exist a mapping $\gamma': F \to [0,1]$ so that $\gamma'(\phi_m) = 1 - \eta(L_{\phi_m})$ $\forall \phi_m \in F$. Here $\gamma'$ is a FGAFS of $F$. Also $\gamma(m) = 1 - \eta(K_m) = 1 - \eta(L_{\phi_m}) = \gamma'(\phi_m) = \gamma' \circ \xi(m)$ . Hence $\gamma$ and $\gamma'$ isomorphic to each other.

**Theorem 4.5.** Every AFS on the basis of $T_1^*$ is SGAFS.

**Proof.** Let $\gamma$ be an AFS of a group $H$. For all $t \in [0,1]$ let $\bar{\gamma}_t$ be the lower level subgroups of $H$. Let $\tau: [0,1] \to \Delta$ be such that $\tau(t) = \bar{\gamma}_t$ where $\Delta$ is the collection of all subgroups of $H$. Let $\mathcal{V}$ represents a $\sigma$-algebra consi-sting of some subsets of $\Delta$ and $\eta$ be the measure on $\Delta$ which is induced by $\tau$ using Lebesgue measure $\eta'$ on [0,1]. Also for any measurable subset $A$ of $\Delta$, $\eta(A) = \eta'(\tau^{-1}(A))$. Let $K_m = \{K \in \Delta : m \in K\}$. Now as $\bar{\gamma}_t$ is a level subgroup of $H$, $\forall t \in [\gamma(m), 1]$ $m \in \bar{\gamma}_t$ . Note that $\bar{\gamma}_t \in K_m$ iff $m \in [\gamma(m), 1]$ and hence $\tau^{-1}(K_m) = [\gamma(m), 1]$. Now $H$ along with $(\Delta, \mathcal{V}, \eta)$ satisfy same co-nditions of Theorem 4.1. Hence $\rho: H \to [0,1]$ can be def-ined as $\rho(m) = 1 - \eta(K_m)$ for all $m \in H$. Here $\rho$ is a SGAFS.

Again, for all $m \in H$

$$\gamma(m) = 1 - \eta'[\gamma(m), 1] = 1 - \eta'(\tau^{-1}(K_m)) = 1 - \eta(K_m) = \rho(m).$$

So, $\gamma = \rho$ is a SGAFS.

**Theorem 4.6.** Let $\gamma$ be a SGAFS and $H$, $K_m$, $(\Delta, \mathcal{V}, \eta)$ represent the same as mentioned in Theorem 4.1. Let there exists a set of subgroups of $H$, denoted as $L$ such that $L \in \mathcal{V}$, $\eta(L) = 0$ and it is simply ordered in terms of set inclusion. Then $\gamma$ is an AFS with on the basis of $T_1^*$.

**Proof.** let $m, u \in H$. As $L$ is simply ordered in terms of set inclusion either $(K_m \cap L) \subseteq (K_u \cap L)$ or $(K_u \cap L) \subseteq (K_m \cap L)$. Let's assume $(K_m \cap L) \subseteq (K_u \cap L)$. Let $K \in (K_m \cap L)$ then $K \in (K_u \cap L)$. So, $m, u \in K$ and as $K$ is a subgroup of $H$, $mu \in K$. Hence $K \in K_{mu}$. So, $(K_m \cap L) \subseteq K_{mu}$.

Now

$$\gamma(mu) = 1 - \eta(K_{mu}) \leq 1 - \eta(K_m \cap L) = 1 - \eta(K_m) \leq T_1^*(1 - \eta(K_m), 1 - \eta(K_u)) = T_1^*(\gamma(m), \gamma(u)).$$

So, $\gamma$ is an AFS on the basis of $T_1^*$.

From Theorem 4.5 and Theorem 4.6, the following can be derived:

**Theorem 4.7.** An AFS is an AFS on the basis of $T_1^*$ iff it is SGAFS and the corresponding generating class carries a subclass of a measure which is simply ordered in terms of set inclusion.

## V.CONCLUSION

We have discussed a new notion of AFS. To the best of our knowledge, this has been introduced for the first time. In Subsection 2.4 with some numerical examples, we have discussed some drawbacks of the existing notion of AFS which can be avoided using our proposed notion. In Section 3 we have defined the infimum image of a fuzzy set and discussed a new proposition. Furthermore, in Section 4 we have defined two families of AFS. One of them is SGAFS and another one is FGAFS. Finally, we have discussed that SGAFS and FGAFS are equivalent to each other and an AFS with respect to $T_1^*$ belong to a special class of SGAFS. Hence, our proposed notion of AFS is superior to any existing one.


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