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Directed Divisor Function Graph $G_{Dij}(n)$

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Abstract— A newer class of graph namely directed divisor function graph is defined and analyzed. Further, directed divisor function sub-digraph, complete directed divisor function graph and characteristics like tournament, Eulerian, Hamiltonian have been discussed.

1. INTRODUCTION

For a positive integer n, the *divisor graph* $\mathbf{G_n}$ having the vertex set as first n natural numbers and any edge is projected between two numbers in which either of them divides the other. Thus in $\mathbf{G_n}$ any vertices i and j are adjacent iff lcm(i, j) = max(i, j) or equivalently gcd(i, j) = min(i, j). Pomerance [1] provided the necessary and sufficient condition for the divisor graph G to a non void positive integers set S as vertices and the adjacency is defined such that for any i, j ∈ S, i and j are adjacent only when gcd(i, j) = min(i, j). Notice that, 1 ≤ gcd(i, j). Also, the concept of a *relatively prime graph* RP (S) for a non empty positive integer set S as its vertex set is defined such that for any two i, j ∈ S, i and j are adjacent only when if they are co-primes. Further properties were studied by Chartrand, Muntean, Saenpholphant and Zhang [2] shown divisor graphs are perfect graphs, a graph G is perfect if every induced subgraph of G has chromatic number equal to the size of a largest clique contained in the subgraph. Le Anh Vinh [3] established the existence of divisor graph of order n for any positive interger n of size m, where m < n. Further, the condition for the cartesian product of two graphs G and H to be a divisor graph was invesitaged by Christopher Frayer. Then, S. Al-Addasi, O. A. AbuGhneim and H. Al-Ezeh [4] showed that divisor graphs cannot have induced odd cycles of length more than three, but there is a possibility for triangles. Singh and Santhosh redefined the notion of a divisor graph in the year 2000. They considered a divisor graph G as an ordered pair (V, E) where V is a subset of the integer set Z such that for any u, v ∈ V, uv ∈ E only when u divides or v divides u, where u ≠ v. Note that, any graph G is isomorphic to a divisor graph is also considered as a divisor graph. The graph of divisor function **D(n)** [5] was introduced by K. Kannan, D. Narasimhan and S. Shanmugavelan in 2015.

In the present paper, the directed divisor function graph is defined and its properties were studied. Further directed divisor function sub-digraph, completeness, connectivity, tournaments in directed divisor function graph, Eulerian, king and directed planarity were discussed. Moreover, every directed divisor function graph $\mathbf{G_{Dij}(n)}$ is a divisor graph G_n. But, the converse need not be true. The directed divisor function graph $\mathbf{G_{Dij}(n)}$ is a digraph having its vertex set as the divisors or factors of n, n ∈ N and any arc can be drawn if we take any pair of divisors either one of them divides the other.

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2. THE DIRECTED DIVISOR FUNCTION GRAPH

The divisor function **D(n)** is defined such that D(n) = {d: d|n, n is a positive integer}.

Defintion 2.1: Fo r any positive integer n ≥ 1 with r divisors $d_1, d_2, \ldots, d_r$ the graph of directed divisor function $\mathbf{G_{Dij}(n)}$ is a graph whose vertex set $\{d_1, d_2, \ldots, d_r\}$ if $d_i|d_j$ then there is an arc from d_i to d_j for all i, j and i ≠ j.

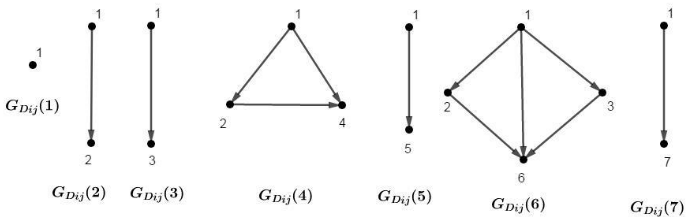


Fig. 1. Directed divisor function graphs G_{Dij} (n) for n = 1 to 7

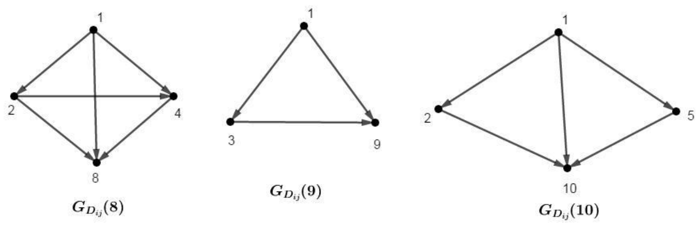


Fig. 2. Directed divisor function graphs G_{Dij} (n) for n = 8 to 10

Note 1: For convenient let us assume $1 = d_1$ and $n=d_r$, where d_i, 1 < i < n are the proper divisors of $\mathbf{G_{Dij}(n)}$, ∀n ∈ Z^+.

Theorem 2.2: **The directed divisor function graph $\mathbf{G_{Dij}}$ (n) is not strongly connected for any n ∈ $\mathbf{Z^+}$.**

Proof: Suppose $\mathbf{G_{Dij}}$ **(n)** is strongly connected then by the definition of strongly connected, for every pair of divisors (d_i, d_j) if there is a d_i - d_j path there must be a d_j − d_i path, which is a contradiction to our definition of $\mathbf{G_{Dij}}$ **(n).**

$\mathbf{G_{Dij}}$ **(n)** is not unilaterally connected in general, the following theorem gives under what condition $\mathbf{G_{Dij}}$ **(n)** is unilaterally connected.

***Theorem 2.3:* G_{Dij} (n) is unilaterally connected if and only if $n = p^a$, where p is prime and $a \in Z^+$.**

Proof:

Sufficient Part: Suppose if n takes the form p^a where p is prime and $a \in Z^+$ then the divisors of n are $\{p^0 = 1, p^1, p^2 \ldots, p^{a-1}, p^a = n\}$. Since $p^0 = 1$ is trivial divisor and it divides all the other divisors of n there is an arc from 1 to all the other divisors of G_{Dij} (n) and let p^i and p^j be be two arbitrary divisors of n, it is clear that $p^i | p^j$ $\forall$ i < j. Thus for every pair of divisors in G_{Dij} (n) either one is reachable from the other and hence $G_{Dij}(n)$ is unilaterally connected.

Necessary Part: Conversely, assume that $G_{Dij}(n)$ is unilaterally connected.

Claim : $n = p^a$, p is prime and $a \in Z^+$.

Suppose $n \neq p^a$, p is prime and $a \in Z^+$ then n can be expressed as $n = p_1^{a1} p_2^{a2} \ldots p_k^{ak}$ where p_i's are distinct $1 \leq i \leq k$. Let $p_1 \neq p_2$ and it is clear that there is no arc between p_1 and p_2 and by the definition of $G_{Dij}(n)$, neither p_1 nor p_2 is reachable from each other which contradicts the fact that $G_{Dij}(n)$ is unilaterally connected. This completes the proof.

Theorem 2.4 Every directed divisor function graph G_{Dij} (n) is weakly connected.

Theorem 2.5 A directed divisor function graph G_{Dij} (n) has a spanning arc sequence whenever $n = p^a$, where p is prime and $a \in Z^+$.

Proof: Since n takes the form p^a, p is prime and $a \in Z^+$, the divisors of n are $\{p^0 = 1, p^1, p^2 \ldots, p^{a-1}, p^a = n\}$ and G_{Dij} (n) is unilaterally connected from theorem 2.3. Further, the arc sequence $p^0 = 1, a_1, p^1, a_2, p^2 \ldots, p^{a-1}, a_n, p^a = n$ is a spanning arc sequence of G_{Dij} (n).

Theorem 2.6 A directed divisor function graph G_{Dij} (n) is unilaterally connected then it contains a directed walk (not closed) containing all the divisors of G_{Dij} (n).

Corollary 2.6.1 For any $n \in Z^+$, the directed divisor function graph G_{Dij} (n) has no cycle.

Theorem 2.7 For any $n \in Z^+$ and n is composite, the directed divisor function graph G_{Dij} (n) is not a directed bipartite graph.

Proof: Let n be a positive composite number. Suppose G_{Dij} (n) is a directed bipartite graph then there exist partitions (V_1, V_2) and the induced directed graphs $D[V_1]$ and $D[V_2]$ contains no arcs of D and there is an arc from $D[V_1]$ to $D[V_2]$ or vice versa. We assume that $1 \in D[V_1]$ and $n \in D[V_2]$..

Since n is not a prime there must exists at least one proper divisor say k such that one divides k and k divides n. Thus there is an arc from 1 to k and an arc from k to n. Therefore, k does not belongs to either $D[V_1]$ and $D[V_2]$ which is a contradiction. Thus , G_{Dij} (n) is not a directed bipartite graph.

Corollary 2.7.1 For any prime p, G_{Dij} (p) is directed bipartite graph.

Theorem 2.8 For any $n \in Z^+$, the directed divisor function graph G_{Dij} (n) is not complete.

Proof: It is clear that from the definition of directed divisor function graph, for any $n \in Z^+$, G_{Dij} (n) is not complete.

Note 2 The necessary and sufficient condition for a underlying graph of $G_{Dij}(n)$ to be complete only when n takes the form p^a, p is prime and $a \in Z^+$.

The following algorithm $G_{Dij}(n)$ construction is used for determining the number of arcs for any $n \in Z^+$.

Algorithm 1 Size of G_{Dij} (n)

INPUT: n (a positive integer)

Step 1: Find all the divisors of n.

Step 2: Calculate the number of divisors.

Step 3: Check the divisibility within the divisors.

Step 4: If $d_i | d_j$ then state that there is an arc from d_i to d_j whenever $i \neq j$.

OUTPUT: The total number of arcs.

Note 3 *The complexity of the algorithm Size of $G_{Dij}(n)$ is O(n).*

Result 1 If ***G_{Dij} (n) is unilaterally connected with k divisors then the total number of arcs is*** $k(k-1)/2$.

Theorem 2.9 (Correctness of Size of G_{Dij} (n)) For any $n \in Z^+$, the algorithm size of G_{Dij} (n) terminates and determines all possible arcs between the divisors of n, $n \in Z^+$.

Proof: This procedure terminates since there are only a finite number of divisors for any positive integer n.

Claim : This procedure determines all possible arcs between the divisors of n.

Suppose G_{Dij} (n) has only one divisor and it is clear that by the definition of G_{Dij} (n), it has no arcs. Suppose G_{Dij} (n) has exactly two divisors and let it be d_1 and d_2. Then it is clear that $d_1 < d_2$ and again by the definition of G_{Dij} (n) if $d_1 | d_2$ then there is an arc from d_1 to d_2.

Assume that the statement is true for all G_{Dij} (n) with k divisors.

Now, consider a directed divisor function graph G_{Dij} (n) with k + 1 divisors. Let the divisors be $d_1, d_2, \ldots, d_k, d_{k+1}$. Remove d_{k+1} from G_{Dij} (n), we have G'_{Dij} (n) = G_{Dij} (n) − $\{d_{k+1}\}$. Then by our assumption, G^0_{Dij} (n) determines all possible arcs.

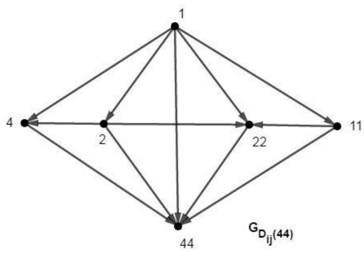
By adding d_{k+1} to G'_{Dij} (n), let d_i be any arbitrary divisor of G'_{Dij} (n) with $i \neq k + 1$, then by the definition of G_{Dij} (n), if any $d_i | d_{k+1}$, then there will be a $d_i d_{k+1}$ arc in G_{Dij} (n). Thus it determines all possible arcs.

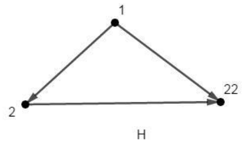
Note 4 Every directed divisor function graph G_{Dij} (n) is not a functional, $\forall$ n $\in Z^+$.

Note 5 In a directed divisor function graph , the shortest path between any two divisors d_i, d_j is one if and only if $d_i | d_j$, $i \neq j$.

Remark 1 A sub-digraph of a directed divisor function graph need not be a directed divisor function graph. The following example supports our claim.

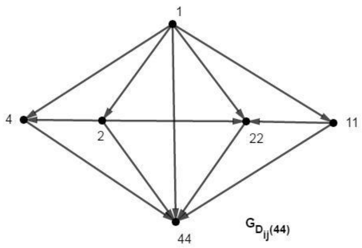
Example 2.10 Consider the graph G_{Dij} (44) and H be a sub-digraph(as given below). Clearly H is not a directed divisor function graph.

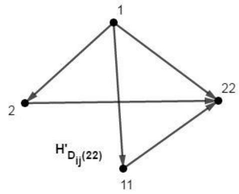




***Definition 2.11** A sub-digraph H of a directed divisor function graph G_{Dij} (n) is said to be a directed divisor function sub-digraph if H itself is a directed divisor function graph.*

Example 2.12 Clearly, **H'_{Dij} (22)** is a directed divisor function sub-digraph of ***G_{Dij} (44),*** *since* **H'_{Dij} (22)** itself is a directed divisor function graph.





Note 6 Every directed divisor function graph G_{Dij} (n) with k divisors has exactly k − 2 proper directed divisor function sub-digraphs.

For any n ∈ Z^+ where n is composite, the directed divisor function graph G_{Dij} (n) is not a directed tree though it is connected acyclic but it does not satisfy the property that if a tree has n vertices it should have n − 1 arcs.

Lemma 2.1 Every directed divisor function graph G_{Dij} (n) has a source and a sink.

Proof: Since 1 is the trivial divisor for any n ∈ Z^+ and if G_{Dij} (n) has k divisors then there exists k − 1 arcs incident out from 1 to all the other divisors of n. For convenient, we take d_1 = 1. No other divisor divides 1, and thus there is no arc incident in to 1 which proves that the in-degree of d_1 is zero which is a source. (i.e.) $d^-(d_1) = 0$. Similarly, we prove n = d_r as a sink. (i.e.) n = d_r.

Result 2 If d_i is the only source of G_{Dij} (n) then for every divisor d_j with i ≠ j, there exists a $d_i − d_j$ path.

Result 3 If d_i is the only source of G_{Dij} (n) then for every divisor d_j with i ≠ j, there exists a $d_j − d_i$ path.

Lemma 2.2 Every directed divisor function graph G_{Dij} (n) has a topological ordering of divisor.

Proof: From the lemma 2.1, G_{Dij} (n) has source (i.e.) $d^-(d_1)$ = 0. Now, remove d_1 from G_{Dij} (n) together with the arcs incident on d_1 from G_{Dij} (n).

The remaining di-graph G_{Dij} (n) − d_1 is still acyclic since G_{Dij} (n) is acyclic. Now, G_{Dij} (n) − d_1 has at least one source.

Let d_2 be the divisor whose in-degree is zero. Next, remove d_2 from G_{Dij} (n) − d_1 which results in G_{Dij} (n) − {d_1, d_2} is also acyclic.

By continuing this process till no divisor(vertex) is of in-degree zero, then the topological ordering of divisors for G_{Dij} (n) is the order of divisors in which they were removed.

***Definition 2.13** Let d_i's (1 ≤ i ≤ r) be the divisors of G_{Dij} (n) and (d_i, d_j) ∈ A(G_{Dij} (n)) and (d_j , d_k) ∈ A(G_{Dij} (n)) ⇔ ((d_i, d_j), (d_j , d_k)) ∈ L(A(G_{Dij} (n))), where d_i is a proper divisor of d_j and d_j is a proper divisor of d_k.*

Remark 2 The following statements are some of the observations of L(G_{Dij} (n)).

1. L(G_{Dij} (n)) is always disconnected.

Proof: Since 1|n and d_r = n which does not divide any of the other divisors and thus there is no divisor such that $d_n|d_k$

∴ (1, d_n) ∈ A(G_{Dij} (n)) and (d_n, d_k) does not belongs to A(G_{Dij} (n)) ∀n.

By the definition of divisor line graph, we have (1, d_n) is an isolated vertex in L(G_{Dij} (n)∀n. Hence L(G_{Dij} (n)) is always disconnected.

2. For a prime n, L(G_{Dij}(n)) has exactly one component(as isolated point).

3. If n has exactly two non-trivial prime divisors then L(G_{Dij} (n)) has exactly three components. Otherwise, L(G_{Dij}(n)) has exactly two components ∀n, but not prime.

4. If G_{Dij} (n) has k divisors then L(G_{Dij} (n)) has at most $e \le \sum_{i=1}^{k} od(di) * id(di)$, where e is the number of edges in L(G_{Dij} (n)).

Result 4 Every directed divisor function G_{Dij} (n) is not Eulerian.

Theorem 2.14 A directed divisor function graph G_{Dij} (n) is said to be a tournament if and only if n = p^a, p be a prime number and a ∈ N.

Proof: Sufficient Part : Assume that a directed divisor function graph G_{Dij} (n) with n = p^a, p is prime and a ∈ Z^+.

Claim : G_{Dij} (n) is a tournament.

It is clear that if n = p^a, p is prime and a ∈ Z^+ then theorem 2.3 provides G_{Dij} (n) is unilaterally connected and thus from every pair of divisors either one is reachable from the other. Thus the underlying graph is complete which results that G_{Dij} (n) is a tournament whenever n = p^a, p is prime and a ∈ Z^+.

Necessary Part : Conversely, assume that G_{Dij} (n) is a tournament. (i.e.) The underlying graph G_{Dij} (n) is complete.

Claim : n = p^a, p is prime and a ∈ Z^+.

Suppose n ≠ p^a, then n = $p_1^{a1} p_2^{a2} \ldots p_m^{am}$, where at least one p_i will be distinct, 1 ≤ i ≤ m. Let p_i and p_j (i ≠ j) be any two distinct divisors and it is clear that p_i and p_j does not divides each other ∀i, j which is a contradiction to our assumption.

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Note 7 We denote the above such tournaments by T_{ij} (n).

Corollary 2.14.1 $G_{Dij}(n)$ contains a directed Hamiltonian path whenever $n = p^a$, p is a prime and $a \in Z$.

Theorem 2.15 The vertex set of G_{Dij} (n) can be ordered such that the adjacency matrix of G_{Dij} (n) is upper triangular.

Proof: Consider the adjacency matrix A of G_{Dij} (n) in which the vertex set can be ordered using the lemma 2.2.

It is clear that G_{Dij} (n) has no self loops. Thus the diagonal entries of A is zero.

Let $d_1, d_2, \ldots, d_r$ be the vertex set of G_{Dij} (n) such that it can be ordered in a way that $d_1 < d_2 < \ldots < d_r$. From the definition of G_{Dij} (n), if $d_i|d_j$ then there is an arc from d_i to d_j which means that the ij^{th} entry of A is 1.

Let $d_1 = 1$, it is clear the first column corresponding to d_1 is zero since no such divisor of n divides 1.

Let d_2 be any prime divisor of n which divides its multiples alone and thus the second column corresponding to d_2 is non-zero only in the first row.(i.e.) $a_{12} \neq 0$ and the remaining entries of second column are zero.

By continuing in this manner, we get the last column corresponding to $d_r = n$ consists 1 all entries except at a_{rr}. Thus, the adjacency matrix A of G_{Dij} (n) is upper triangular.

Theorem 2.16 T_{ij} (n) is transitive if and only if the divisor score sequence of G_{Dij} (n) is 0, 1, . . . , r − 1.

***Proof:* Sufficient Part :** Consider the sequence S : 0, 1, 2, . . . , r − 1. Let T_{ij} (n) be the tournament with r divisors.

claim : T_{ij} (n) is transitive.

Consider $V(T_{ij}(n)) = \{d_0, d_1, \ldots, d_{r-1}\}$ where each d_i's are the divisors of n and an arc set $A(T_{ij}(n)) = \{(d_i, d_j)|0 \leq i, j \leq r - 1\}$ and $i \neq j$. and the divisor score sequence is distinct it is clear that T_{ij} (n) is acyclic which leads that T_{ij} (n) is transitive.

For $0 \leq i \leq m$, od $d_i = r - i$.

Necessary Part : Conversely, Assume that T_{ij} (n) is a tournament of order r and it is transitive.

claim : 0, 1, 2, . . . , r − 1 is the score sequence of T_{ij} (n).

It is enough to show that no two divisors have same scores.

Let $d_i, d_j \in T_{ij}$ (n). Since $n = p^a$ it is either $d_i|d_j$ or $d_j|d_i$. Assume that $d_i|d_j$.

Consider the set $W = \{d_k : d_j |d_k\}$ then $(d_j , d_k) \in A(T_{ij}$ (n)). Therefore, the divisors of n that are divisible by d_j equals |W |. Since T_{ij} (n) is transitive we have $d_i|d_k$. It implies that od $d_i \geq |W| + 1$.

(i.e) od $d_i \geq$ od $d_j + 1$.

Therefore, score of $d_i \neq$ score of d_j , $\forall i \neq j$.

Hence the score sequence of T_{ij} (n) of order r is 0, 1, 2, ..., r-1.

Similarly, we prove $d_j |d_i$.

Theorem 2.17 Every directed divisor function graph G_{Dij} (n) has a king.

Proof: Let $G_{Dij}(n)$ be a directed divisor function graph and it is clear that 1 is a divisor of all n. Thus there is an arc from 1 to all other divisors in G_{Dij} (n) which means that 1 is the king of G_{Dij} (n) $\forall n \in Z^+$.

Definition 2.18 The directed divisor function graph G_{Dij} (n) is said to be quasi strongly connected if for every pair of divisors d_i and d_j of n, there is a divisor d_k such that $d_k|d_i$ and $d_k|d_j$.

Note 8 For any pair of divisors (d_i, d_j) of G_{Dij} (n), the trivial divisor $d_k = 1$ divides both d_i and d_j $\forall$ n.

Definition 2.19 The divisor d_i of G_{Dij} (n) is said to be a root if d_i divides all the other divisors of n in G_{Dij} (n).

Result 5: *1 is the root of all G_{Dij} (n), $\forall n \in Z^+$.*

Note 9 No other divisors will be the root of G_{Dij} (n), since no other divisor will divide 1.

In general, not all G_{Dij} (n) are directed planar, the following theorem states that under what condition G_{Dij} (n) will be directed planar.

Theorem 2.20 The directed divisor function graph G_{Dij} (n) with $n = p^a$ is directed planar if it has at most 4 divisors in it.

Proof: We know that for a simple directed graph without loops and with at least 3 vertices we have $a \leq (3*n) - 6$.

From this it is clear that if $n \leq 4$ the result is true. Suppose $n > 4$ the inequality does not holds.

3. CONCLUSION AND FUTURE WORK

Thus, the nature of directed divisor function graph have been studied and it is clear that every $G_{Dij}(n)$ is a Directed Acyclic Graph. Moreover, the size of the $G_{Dij}(n)$ was obtained by implementing an algorithm. Several results concerning connectedness, completeness, sub-digraph, tournament, line digraph, Hamiltonian, king, root and planarity associated with them were discussed. Further research on finding the factors of any disease can be undertaken with the help of this directed divisor function graph.

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