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Connectivity, Independency and Colorability of Divisor Function Graph $G_{D(n)}$

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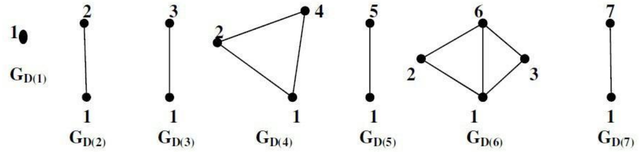
Abstract— ***The aim of the present paper is to find the connectivity of divisor function graph and to determine the colorability of the divisor function graph via Independency of the divisor function graph. Further, the condition for divisor function graph to be isomorphic have been discussed.***

1. INTRODUCTION

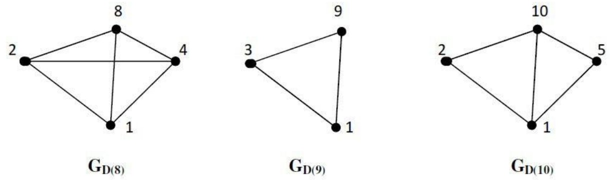
Pomerance [1] extended the definition of the divisor graph to any non empty set S of positive whose divisor graph G(S) has vertex set S and any two vertices i and j are adjacent iff gcd(i, j) = min(i, j). Certainly, $1 \leq \gcd(i, j)$. If equality of this considered, it makes him to define a relatively prime graph RP (S) of S having vertex set as S and any two vertices are adjacent iff they are co-primes. Further he showed that every graph is a relatively prime graph. Further properties were studied by Chartrand, Muntean, Saenpholphant and Zhang [2] showed that divisor graphs are perfect graphs. Singh and Santhosh redefined the notion of a divisor graph in the year 2000. They considered a divisor graph G as an ordered pair (V, E) where V is a subset of the integer set Z such that for any u, v ∈ V, uv ∈ E only when u divides v or v divides u, where u ≠ v. In 2015, the graph of divisor function D(n) [3] was introduced by K. Kannan, D. Narasimhan and S. Shanmugavelan. In this paper the concept of connectedness and colourings in divisor function graph are given. The criteria for 2 connected and theorems on 4 connected have been discussed. Also necessary and sufficient condition for perfect matching in divisor function graph is given. Also chromatic number and the chromatic index of divisor function have been introduced.

2 CONNECTIVITY OF DIVISOR FUNCTION GRAPH $G_{D(n)}$

Definition 2.1 For any positive integer $n \geq 1$ with r divisors $d_1, d_2, d_3, \ldots, d_r$ the graph of divisor function D(n) of n is a graph $G_{D(n)}$ with the vertex set $\{d_1, d_2, d_3, \ldots, d_r\}$ such that two vertices d_i and d_j are adjacent iff either $d_i | d_j$ or $d_j | d_i$, $i \neq j$.



Divisor function graphs $G_{D(n)}$ for n = 1 to 7



Divisor function graphs $G_{D(n)}$ for n = 8 to 10

Definition 2.3 Two divisor function graphs $G_{D(n_1)}$ and $G_{D(n_2)}$ are said to be isomorphic if n_1 and n_2 have same number of divisors and same number of prime factors.

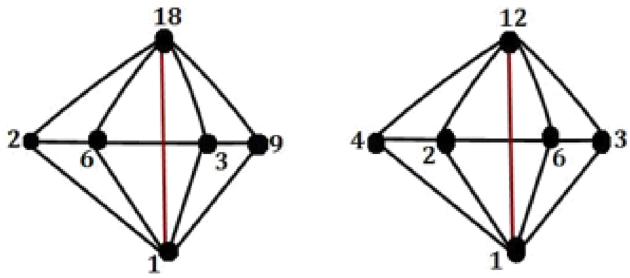


Figure 2: $G_{D(18)}$ and $G_{D(12)}$

Remark: $G_{D(p1)} \approx G_{D(p2)}$ where p1 and p2 are prime numbers.

Note: For convenient we say vertices as divisors.

Definition 2.4 A semiprime is a composite number that can be expressed as a product of two (possibly equal) primes.

Example 2.5 { 4, 6, 9, 10, 14, 15, 21, 22.}

Theorem 2.6 $G_{D(n)}$ is two connected if and only if n is a semi-prime such that n is not a perfect square.

Proof: Let n be a semi-prime such that n is not a perfect square. Then $G_{D(n)}$ has exactly four divisors and let it be $1 = d_1$, d_2, d_3 and $d_4 = n$ where d_2 and d_3 are prime and hence they are not adjacent in $G_{D(n)}$. Since 1 and n are adjacent to both d_2

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and d_3, we have $G_{D(n)} - d_1$ and $G_{D(n)} - d_4$ are connected graph, but $G_{D(n)} - \{1, n\}$ leads to a disconnected graph.

Hence $G_{D(n)}$ is 2-Connected.

Conversely, assume that $G_{D(n)}$ with r divisors is 2-connected. $G_{D(n)} - \{1, n\}$ is disconnected since 1 and n are the only divisors that are adjacent to all proper divisors of $G_{D(n)}$.

Claim : n is a semi-prime such that n not a perfect square.

Suppose not n is a semi-prime, then there are 2 cases:

(i) If n has exactly one prime then $G_{D(n)}$ is complete and hence it contradicts our hypothesis.

(ii)If n has 2 or more prime divisors.

Let $p_1, p_2, \ldots, p_s$ be the prime divisors of n and $k_1, k_2, \ldots, k_l$ be the composite divisors of n. Since each k_i is divisible by some p_j' s, each k_i is adjacent to p_j for some j, where $1 \le i \le l$ and $1 \le j \le s$. Also each pair (p_i, p_j) has a composite k_m such that $p_i|k_m$ and $p_j|k_m$ for some m, $1 \le m \le l$.

Hence $G_{D(n)} - \{1, n\}$ remains connected which is a contradiction to our assumption.

{we consider 1 and n since they are adjacent to all the other divisors}

Suppose n is a semi-prime. There are 2 cases:

1. If n is a perfect square, then $G_{D(n)}$ is complete and hence it is not 2-connected.

2. If n is not a perfect square then there are exactly 2 proper divisors p_1 and p_2 such that $p_1 \neq p_2$ and so they are not adjacent.

Hence $G_{D(n)} - \{1, n\}$ is disconnected.

For example consider $G_{D(21)}$

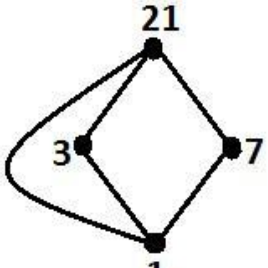
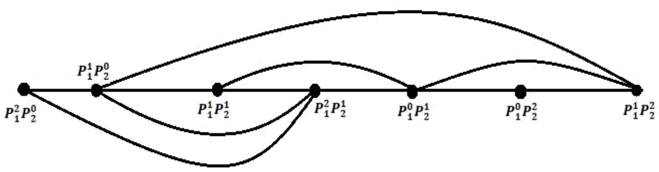


Figure 3: $G_{D(21)}$



Theorem 2.7 If n is a perfect square such that $n = p_1^2 p_2^2$ where p_1 and p_2 are distinct primes then $G_{D(n)}$ is 4-connected.

Proof: Let n be a perfect square with $n = p_1^2p_2^2$ where p_1 and p_2 are distinct primes. Then $V(G_{D(n)}) = \{p_1^a p_2^b;\ 0 \le a, b \le 2\}$.

We claim that $G_{D(n)}$ is 4-connected.

Since $p_1^0p_2^0 = 1$ and $p_1^2p_2^2 = n$ are adjacent to all the other divisors of n it is enough to prove that $G_{D(n)} - \{1, n\}$ is 2 connected. For convenient we choose $G'_{D(n)} = G_{D(n)} - \{1, n\}$.

By Theorem 2.6, we have $G'_{D(n)}$ is connected. Since there is no vertex (divisor) $p_1^a p_2^b$ whose degree is one we have $G^0_{D(n)} - \{p_1^a p_2^b\}$ is connected. Since $p_1|p_1^2$ and $p_1^2|p_1^2p_2$, p_2^1 is adjacent to both p_1 and $p_1^2p_2$.

Also they are the only adjacent divisors to p_1^2 in $G'_{D(n)}$ and hence $\deg(p_1^2) = 2$. Similarly $\deg(p_2^2) = 2$.

Therefore $G'_{D(n)} - \{p_1, p_1^2p_2\}$ and $G'_{D(n)} - \{p_2, p_1p_2^2\}$ is a disconnected graph. Hence $G'_{D(n)}$ is 2- connected and thus $G_{D(n)}$ is 4 -connected.

Definition 2.8 A sphenic number is a positive integer n which is the product of exactly three distinct primes.

In particular, if p, q and r are prime numbers, then every sphenic number n = pqr has precisely eight positive divisors, namely 1, p, q, r, pq, qr, pr and n itself.

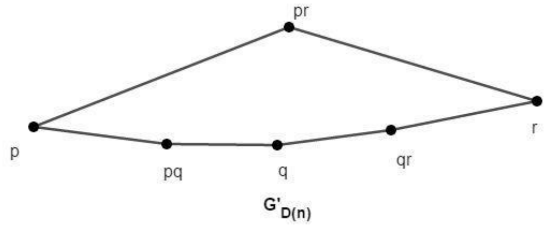
For example consider 30 which is the product of three distinct primes 2,3 and 5. The first few sphenic numbers are 30, 42, 66, 70, 78, 102, 105, 110, 114, ...

Theorem 2.9 If n is a sphenic number then $G_{D(n)}$ is 4-connected.

Proof: Consider $G_{D(n)}$ where n is sphenic. Then n = pqr where p, q, r are three distinct primes and $V(G_{D(n)}) = \{1 = p^0q^0r^0, p, q, r, pq, qr, pr, pqr = n\}$.

We claim that $G_{D(n)}$ is 4 connected.

Since 1 and n are adjacent to all the other divisors of n it is enough to show that $G'_{D(n)} = G_{D(n)} - \{1, n\}$ is 2-connected. By theorem 2.6 it is clear that $G'_{D(n)}$ is connected.



From the above graph of $G'_{D(n)}$ we have that every vertices (divisors) of $G'_{D(n)}$ has a degree 2 and hence it is 2-connected. Therefore, $G_{D(n)}$ is 4-connected.

Remarks:

- A vertex cut of $G_{D(n)}$ is analogous to a simple graph.
- For a composite n, $G_{D(n)}$ has no cut vertex and for prime n, 1 and n are cut vertices.
- In $G_{D(n)}$ if all the proper divisors of n are relatively prime then |S| is 2, where S is the vertex cut.
- In $G_{D(n)}$ if no two divisors of n are relatively prime then there is no cut vertex.

Definition 2.10 A subset S of $V(G_{D(n)})=\{d_1, d_2, d_3, \ldots, d_r\}$ is said to be an independent set if every pair of divisors of S does not divide each other.

Definition 2.11 A subset M of $E(G_{D(n)})$is called edge independent or Matching of $G_{D(n)}$ if d_id_j , $d_kd_l \in M$ such that $i \neq j \neq k \neq l$ for all n.

- Maximum independent set of $G_{D(n)}$ is analogous to simple graph G.

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- {1} and {n} are Maximal independent sets for all $G_{D(n)}$.
- If $G_{D(n)}$ is complete then each $\{d_i\}$is maximal independent set.
- Maximal Matching of $G_{D(n)}$ is analogous to simple graph G.

3. THEORTICAL RESULTS

Theorem 2.12 $G_{D(n)}$ has a perfect matching if and only if n is not a perfect square.

Proof: Consider $G_{D(n)}$ with k divisors that has a perfect matching M. Then M saturates all the vertices of $G_{D(n)}$. We claim that n is not a perfect square.

Suppose n is a perfect square, then by the theorem "If n > 1, then n is a perfect square if and only if n has odd number of divisors" $G_{D(n)}$ has

odd number of divisors which means that k is odd, which is a contradiction to the fact that M is a perfect matching of $G_{D(n)}$.Therefore n is not a perfect square.

Conversely, assume that n is not a perfect square. Let $G_{D(n)}$ has even number(say r) of divisors. Let M be the maximum matching of $G_{D(n)}$.

We will prove that M is a perfect matching. There are two cases:

(i) If n is semi prime then r=4. We choose $1=d_1, d_2, d_3, d_4 = n$. Since d_2 and d_3 are prime they are not adjacent. Thus, $M=\{d_1d_2, d_3d_4\}$ saturates all the vertices of $G_{D(n)}$ and hence M is a perfect matching.

(ii) Consider n which is not a semi prime. For convenient we choose $G_{D(n)} - \{1, n\}$ as $G'_{D(n)}$, $d_1 = 1$ and $d_r = n$. By Theorem 2.6 we have $G'_{D(n)}$ as a connected graph. Consider the spanning subgraph of $G'_{D(n)}$ such that $G'_{D(n)}-d_id_j$ leads to disconnected graph for some $1 < i, j < r$.

Since r is even $G'_{D(n)}$ has even number of proper divisors $d_2, d_3, \ldots, d_{r-2}, d_{r-1}$ and also there are the only two pendent vertices.

It is clear that the edges $d_2d_3, d_4d_5, \ldots, d_{r-2}d_{r-1}$ are independent(not necessarily adjacent in the same order).

Thus, $M =\{d_1d_r, d_2d_3, d_4d_5, \ldots, d_{r-2}, d_{r-1}\}$ saturates all the vertices of $G_{D(n)}$, and hence M is a perfect matching.

Theorem 2.13 The chromatic number of $G_{D(n)}$ is k if k is the least number of independent sets $S_1, S_2, \ldots, S_k$ such that each S_i is the maximum independent set of $G^{i}{}_{D}{}^{-}({}^{1}{}_{n)}$ where $V(G^i_{D(n)}) = V(G_{D(n)}) - \cup_i S_i$ and $S_k = \{1\}$ or $\{n\}$, $1 \le i \le k-1$ and $G'_{D(n)} = G_{D(n)}$.

Proof: Let $G_{D(n)}$ be the divisor function graph with the vertex set $V = \{1 = d_1, d_2, \ldots, d_r = n\}$.

Let S_1 be the maximum independent set of $G_{D(n)}$ that contains only prime or only composite divisors. Otherwise the number of S_i's constructed by the following procedure will be greater than the chromatic number.

For example consider $G_{D(12)}$.

If $S_i = \{3, 4\}$ then it implies that the chromatic number $G_{D(12)}$ is 5 where the actual chromatic number of $G_{D(12)}$ is 4.

It is clear that each $d_i \in S_1$ can be colored by a unique color say C_1. Consider the resultant graph $G^1_{D(n)}$ which is obtained by the removal of S_1 from $V(G_{D(n)})$.

$$\mathbf{V(G^1_{D(n)}) = V(G_{D(n)}) - S_1.}$$

Let S_2 be the maximum independent set of $G^1_{D(n)}$ that contains only prime or composite divisors and hence S_2 can be colored by C_2.

Now, the resultant graph obtained by removing S_2 from $V(G^1_{D(n)})$ be $G^2_{D(n)}$ where $V(G^2_{D(n)}) = V(G^1_{D(n)}) - S_2$.

On continuing this way, we get

$$\mathbf{V(G^{k-2}_{D(n)}) = V(G^{k-3}_{D(n)}) - S_{k-2}.}$$
$$\mathbf{V(G^{k-2}_{D(n)}) = \{1, n\}.}$$

Hence, if $S_{k-1} = \{n\}$ then $V(G^{k-1}_{D(n)}) = \{1\}$ and $S_k = \{1\}$ and if $S_{k-1} = \{1\}$ then $V(G^{k-1}_{D(n)}) = \{n\}$ and $S_k = \{n\}$.

However, S_{k-1} and S_k must have distinct colors C_{k-1} and C_k respectively.

Hence $G_{D(n)}$ is colored by k colors.

Also, $V(G^1_{D(n)}) = V(G_{D(n)}) - S_1$ $V(G^2_{D(n)}) = V(G^1_{D(n)}) - S_2$

$V(G^{k-1}_{D(n)}) = V(G^{k-2}_{D(n)}) - S_{k-1}$

$= V(G_{D(n)}) - S_1 - S_2 - \cdots - S_{k-1}$

$= V(G_{D(n)}) - (S_1 + S_2 + \cdots + S_{k-1})$.

By the addition operation on simple graph, we have $S_1 + S_2 + \cdots + S_{k-1} = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_{k-1}$. Thus, $V(G^i_{D(n)}) = V(G_{D(n)}) - \cup_i S_i$, where $1 \le i \le k-1$ and $S_k = \{1\}$ or $\{n\}$.

Example 2.14 Consider $G_{D(846)}$ which has 12 divisors.

$V(G_{D(846)}) = \{1, 2, 3, 6, 9, 18, 47, 94, 141, 282, 423, 846\}$

$\mathbf{S_1 = \{6, 9, 94, 141\}}$

$V(G^1_{D(846)}) = V(G_{D(846)}) - S_1$.

$=\{1, 2, 3, 18, 47, 282, 423, 846\}$.

$\mathbf{S_2 = \{18, 282, 423\}}$

$V(G^2_{D(846)}) = V(G^1_{D(846)}) - S_2$.

$=\{1, 2, 3, 47, 846\}$.

$\mathbf{S_3 = \{2, 3, 47\}}$

$V(G^3_{D(846)}) = V(G^2_{D(846)}) - S_3$.

$=\{1, 846\}$.

$\mathbf{S_4 = \{1\}}$

$V(G^4_{D(846)} = V(G^3_{D(846)}) - S_4 = \{846\}$

$\mathbf{S_5 = \{846\}}$

Hence $G_{D(846)}$ is 5 colorable.

Theorem 2.15 If $G_{D(n)}$ has r divisors then $\chi'(G_{D(n)}) \geq (r - 1)$.

Proof: Consider $G_{D(n)}$ with r divisors, with $1 = d_1, d_2, \ldots, d_r = n$. Since there are $r-2$ proper divisors and 1 is adjacent to all the proper divisors, each edge $1 - d_i$ must be colored distinctly by $r-2$ colors, $2 \le i \le r-1$. Let the edge $1 - d_i$ is colored by C_i.

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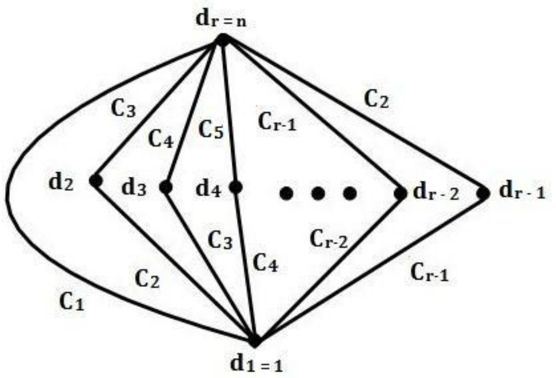

Similarly, each edge $d_i - n$ can be colored properly by these $r - 2$ colors in such a way that each $d_2 - n$ is colored by C_3, $d_3 - n$ is colored by C_4 and so on. On continuing this way we get $d_{r-2} - n$ is colored C_{r-1} and we color $d_{r-1} - n$ by C_2. {since C_2 is not used for any $d_i - 1$ we consider C_2} Hence we colored the edges $1-d_i$ and $d_i - n$ by $r-2$ colors, where $2 \le i \le r-1$. Also the edge $1 - n$ must be colored by C_{r-1} which is distinct from these $r - 2$ colors.

Since there may be more edges in $G_{D(n)}$ than what we considered and we have used $r - 1$ colors, the edges of $G_{D(n)}$ must be colored at least by $r - 1$ colors.

Hence $\chi'(G_{(D(n)}) \ge r - 1$.

Remarks:

1. If n is a semi prime such that n is not a perfect square then $\chi'(G_{(D(n)}) = r - 1$.
2. $\chi'(G_{(D(n)}) = 3$ for all n is of kind (1).
3. If n is a semi prime such that n is a perfect square then $\chi'(G_{D(n)}) = 3$.

Theorem 2.16 Let $G_{D(n)}$ has r divisors where $n = p^a$ and $a \in N$. Then

$\chi'(G_{D(n)}) = r$, if n is not a perfect square

r-1, if n is a perfect square.

Proof: Consider $G_{D(n)}$ with r divisors where $n = p^a$ and n is not a perfect square. Then the divisors are $\{1 = p^0, p^1, p^2, \ldots, p^{r-1} = n\}$. Since n is not a perfect square, r is even and hence $r - 1$ is odd. We will prove that $\chi(G^0_{D(n)}) = r - 1$.

The edges of $G_{D(n)}$ is the form p^ip^j , where $0 \le i < j \le r-1$. We construct the maximum matchings $M_1, M_2, \ldots, M_k$ such that $M_1 \cap M_2 \cap M_3 \cap \ldots \cap M_k = \varphi$. By theorem 2.12, each M_l's are perfect matching and hence $|M_l| = r/2$, $1 \le l \le k$. Since $n = p^a$ and each M_l^0s are perfect matching we have $k = r - 1$.

$|M_1| = r/2$

$|M_2| = r/2$

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$|M_{r-1}| = r/2.$

Hence, all the edges p^ip^j of $G_{D(n)} \in M_1 \cup M_2 \cup \ldots \cup M_{r-1}$, where $1 \le i < j \le r - 1$.

Therefore, each M_k can be colored distinctly by C_k colors, $1 \le k \le r - 1$. Since there are $(r - 1)$ maximum matchings, $G_{D(n)}$ must have $(r - 1)$ colors.

Hence, $\chi'(G_{D(n)}) = r - 1$.

Consider n which is a perfect square. We prove the statement by removing the divisor $d_r = n$ from $G_{D(n)}$, the divisor function graph which we have discussed for perfect square n. Hence $G_{D(n)}$ has odd number of divisors say k(= r−1). Also $|M_i| = (r/2)-1$, where $0 \le i < j \le k$. Hence, all the edges p^ip^j of $G_{D(n)} \in M_1 \cup M_2 \cup \ldots \cup M_{r-1}$, where $1 \le i < j \le r - 1$. Thus, $\chi'(G_{D(n)}) = k$ where $G_{D(n)}$ has k divisors which is odd.

Example 2.17

1. Consider non perfect square n where $G_{D(p^5)}$ has $V(G_{D(p^5)}) = \{p^0, p^1, p^2, p^3, p^4, p$ and $E(G_{D(p^5)}) = \{p^0p^1, p^0p^2, p^0p^3, p^0p^4, \ldots, p^3p^5, p^4p^5\}$.

Let us construct the maximum matchings (necessarily a perfect matchings) as in previous theorem,

$M_1 = \{p^0p^1, p^2p^3, p^4p^5\}$

$M_2 = \{p^0p^2, p^1p^4, p^3p^5\}$

$M_3 = \{p^0p^3, p^2p^4, p^1p^5\}$

$M_4 = \{p^0p^4, p^2p^5, p^1p^3\}$

$M_5 = \{p^0p^5, p^1p^2, p^3p^4\}$.

It is clear that each M_i can be colored by C_i, where $M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5 = E(G_{D(n)})$, $1 \le i \le 5$.

2. Consider perfect square n where $n = p^4$ and $G_{D(p^4)}$ has $V(G_{D(p^4)}) = \{p^0, p^1, p^2, p^3, p^4\}$ and $E(G_{D(p^4)}) = \{p^0p^1, p^0p^2, p^0p^3, p^0p^4, \ldots, p^3p^4, p^3p^4\}$. From (1) we have $M_1 = \{p^0p^1, p^2p^3\}$, $M_2 = \{p^0p^2, p^1p^4\}$, $M_3 = \{p^0p^3, p^2p^4\}$, $M_4 = \{p^0p^4, p^1p^3\}$, $M_5 = \{p^1p^2, p^3p^4\}$.

It is clear that each M_i can be colored by C_i, where $M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5 = E(G_{D(p^4)})$, $1 \le i \le 5$.

Note:

1. From the above discussion it is clear that if $n = p^a$ and n is not a perfect square then there are a such maximum matchings and hence $\chi'(G_{D(n)}) = a$. and if n is a perfect square then $\chi'(G_{D(n)}) = a + 1$.

2. Let $G_{D(n)}$ has r number of divisors and M be the maximum matching of $G_{D(n)}$. If r odd then then $|M| = (r/2) - 1$ and if r even then $|M| = (r/2)$. Hence for all $G_{D(n)}$ $|M|$ is either $(r/2)$ or $(r/2) - 1$.

4. CONCLUSION

Divisor function, the concept of Number Theory had been implemented as graph theoretic view, called divisor function graph $G_D(n)$. Then, the criteria for two connected in $G_D(n)$ had been identified and also by using perfect squares and sphenic number theorems on 4-connected is discussed. The Definitions of vertex independent set and edge independent set were newly stated in $G_D(n)$ by using the divisibility concept of Number theory. If the simple graph has even number of vertices it is not necessary to have perfect matching. But in the case of divisor function graph when the number of divisors(vertices) are even the graph must have necessarily have perfect matching. The procedure for coloring of $G_D(n)$ is analyzed and also its chromatic number is studied by constructing independent sets of the graph.

The lower bound for chromatic index is identified and chromatic index for complete $G_D(n)$ is discussed by using maximum matching. Further the connectivity of $G_D(n)$ may be generalized by finding the degree of maximum powers of prime divisors of n.

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5. ACKNOWLEDGEMENTS

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