

# The Mertens Function and The Proof of The Riemann's Hypothesis

M. Sghiar

**Abstract :** I will prove that  $M(n) = O(n^{1/2+\epsilon})$  where  $M$  is the Mertens function, and I deduce a new proof of the Riemann's hypothesis.

**Keywords:** Prime Number, number theory, distribution of prime numbers, the law of prime numbers, the Riemann hypothesis, the Riemann zeta function, the Mertens function.

## I. INTRODUCTION

### A. Recall, Notations and Definitions

The Riemann's function  $\zeta$  (see [2]) is a complex analytic function that has appeared essentially in the theory of prime numbers. The position of its complex zeros is related to the distribution of prime numbers and is at the crossroads of many other theories. The Riemann's hypothesis (see [5] and [6]) conjectured that all nontrivial zeros of  $\zeta$  are in the line  $x = \frac{1}{2}$ .

The Möbius function generally designates a particular multiplicative function, defined on the strictly positive integers and with values in the set  $\{-1, 0, 1\}$ :

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has at least one repeated factor} \\ 1 & \text{if } n=1 \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct prime factors.} \end{cases}$$

In number theory, the function of Mertens is defined by :  $M(n) = \sum_{1 \leq k \leq n} \mu(k)$  and it has been falsely conjectured by Mertens that the absolute value of  $M(n)$  is always less than  $\sqrt{n}$ , and that if we can prove that the absolute value of  $M(n)$  is always less than  $\sqrt{n}$ , so Riemann's hypothesis is true.

This conjecture is passed for real for a long time. But in 1984, Andrew Odlyzko and Hermante Riele [1] show that there is a number greater than  $10^{30}$  which invalidates it (See also [3] and [4]) However it has been shown that the Riemann's hypothesis [2] is equivalent to the following conjecture:

**Conjecture :**  $M(n) = O(n^{1/2+\epsilon})$

The purpose of this paper is to prove this conjecture and to deduce a new proof of Riemann's hypothesis. Recall also that in [7] it has been shown that prime numbers are defined by a function  $\Phi$ , follow a law in their appearance and their distribution is not a coincidence.

**Manuscript published on 30 December 2017.**

\* Correspondence Author (s)

M. Sghiar,\* Allée Capitaine J. B Bossu, 21240, Talant, France. E-mail: [msgghiar21@gmail.com](mailto:msgghiar21@gmail.com)

© The Authors. Published by Blue Eyes Intelligence Engineering and Sciences Publication (BEIESP). This is an open access article under the CC-BY-NC-ND license <http://creativecommons.org/licenses/by-nc-nd/4.0/>

## II. THE PROOF OF THE CONJECTURE

**Proposition 1 :**  $M(n) = O(n^{1/2+\epsilon})$

**Proof :** Let  $\epsilon$  such that  $0 < \epsilon < \frac{1}{2}$ .

Note that we have one of the possibilities :

- i-  $M(n+1) = M(n)$
- ii-  $M(n+1) = M(n)+1$
- iii  $M(n+1) = M(n)-1$

In any case we have:

$$\left| \frac{M(n+1)}{(n+1)^{1/2+\epsilon}} \right| \leq \left| \frac{M(n)}{n^{1/2+\epsilon}} \right| + \frac{1}{(n+1)^{1/2+\epsilon}}$$

It follows that:

$$\left| \frac{M(n+1)}{(n+1)^{1/2+\epsilon}} \right| \leq \sum_{k=1}^{n+1} \frac{1}{k^{1/2+\epsilon}} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{1/2+\epsilon}} - \sum_{k=n+2}^{+\infty} \frac{1}{k^{1/2+\epsilon}}$$

And hence for  $n$  large enough we have:

$$\left| \frac{M(n+1)}{(n+1)^{1/2+\epsilon}} \right| \leq \left[ (1/2+\epsilon) \right] + o(1)$$

And the result is deduced.

**Corollary 1** (The Riemann hypothesis) :

All non-trivial zeros of  $\zeta$  are in the line  $x = \frac{1}{2}$ .

**Preuve :** By posing  $M(x) = \sum_{1 \leq k \leq x} \mu(k)$ , it is known that:

$$\frac{1}{\zeta(z)} = z \int_1^{\infty} \frac{M(x)}{x^{z+1}} dx$$

The Proposition 1 shows that this integral converges for  $\Re(z) > 1/2$ , implying that  $1/\zeta$  is defined for  $\Re(z) > 1/2$  and by symmetry for  $\Re(z) < 1/2$ .

# The Mertens Function and The Proof of The Riemann's Hypothesis

Thus, the only non-trivial zeros of  $\zeta$  satisfy  $\sigma(z) = 1/2$ , which is the statement of the Riemann's hypothesis.

## III. CONCLUSION

To prove the Riemann's hypothesis we needed to prove and use a property of the Mertens function (Proposition 1). And curiously and inversely for the proof of this last proposition we have used properties of the Riemann function, which shows the close link between the Mertens function and the Riemann function, as shown by the equations (2), (3) and (4).

## REFERENCES

1. Odlyzko et H. J. J. te Riele, « *Disproof of the Mertens conjecture* », *J. reine angew. Math.*, 357, 138-160.
2. H.L. Montgomery – The pair correlation of zeros of the zeta function, *Proc. of Symposia in Pure Math.*, vol. 24, American Mathematical Society, 1973, p. 181–193.
3. T. Kotnik et J. van de Lune, « *On the order of the Mertens function* », *Experimental Mathematics*, 13, 2004), 473-481.
4. T. Kotnik et Herman te Riele, « *he Mertens Conjecture Revisited* », dans *Proceedings of the 7th Algorithmic Number Theory Symposium*, «Lecture Notes in Computer Science » (4076), 156-167.
5. M. Sghiar, (Décembre 2015), Des applications génératrices des nombres premiers et cinq preuves de l'hypothèse de Riemann, *Pioneer Journal of Algebra, Number Theory and its Applications*, Volume 10, Numbers 1-2, 2015, Pages 1-31.
6. M. Sghiar, (Livre) Cinq preuves de l'Hypothèse de Riemann, Éditions Universitaires Européennes, ISBN-13 :978-3-639-54549-4.
7. M. Sghiar "Découverte de la loi cachée pour les nombres premiers." *IOSR Journal of Mathematics (IOSR-JM)* 13.4 (2017): 48-50.