Finite Time Sliding Mode Controller based on Reduced-Order Observer for the Mismatched Uncertain Systems with a Time Delay

Yao-Wen Tsai, Cong-Trang Nguyen

Abstract*: This paper presents the design of the finite time sliding mode controller based on reduced order observer for time-delay systems with mismatched uncertainties. The main achievements of work are: (1) a suitable reduced order observer (ROO) is constructed to estimate the unmeasurable state variables, (2) a finite time sliding mode controller (FTSMC) is designed by employing the estimated variables, and (3) by the application of the Lyapunov stability theory and the linear matrix inequality (LMI) technique, the stability of the overall closed-loop mismatched uncertain systems with a time delay is guaranteed in sliding mode under sufficient condition. Finally, the design procedure is given to summarize the proposed method.

Index Terms*: Variable Structure Control (VSC), reduced-order observer (ROO), finite-time convergence, mismatched uncertainty, time-varying delay.

I. INTRODUCTION

Generally speaking, most common time-delay is encountered in physical control systems. It frequently induces system instability and bad performance; hence, one of the efficient methods is use the variable structure control (VSC) theory [1], [2]. The VSC has been successfully applied for the stability of the time delay uncertain system with a large number of quality papers published in the most recently internationally renowned journals and the related references therein [3]-[6]. The VSC has various attractive features such as finite-time convergence, fast dynamic response, good robustness, exogenous perturbations rejection ability, and its insensitivity to parameter variations. The VSC theory has been effectively applied to a wide variety of practical time-delay systems such as hydraulic/pneumatic, data transmission, satellite systems, robotic manipulator, chemical processes, communication and network system, etc [7]-[10].

Based on the results of the existing studies, it can see that the traditional method of VSC uses full-state feedback or full-order observer (FOO) which requires extra dynamics to estimate the unmeasurable state variables with large dimension. In some cases, this could not be feasible. In addition, the FOO increases the computation and structure complexity. This leads to develop an output feedback sliding mode controller (SMC) which employs the output information only or reduced-order observer (ROO) that the conservatism is reduced, and the robustness is enhanced in comparison with FOO. To estimate unmeasurable states of systems, the FOO/ROO is investigated by authors [11]-[14]. In [11], the state parameter observer was proposed for multiple-input- multiple output systems with time-varying delay under matching condition. The FOO was established in [12] for a class of time delay uncertain systems with nonlinear disturbance which satisfies the Lipschitz condition. Based on the average dwell-time concept, an observer scheme was studied in [13] for some classes of switched linear systems with unknown inputs. Nevertheless, the error of this observer converges exponentially to zero as time tends to infinity. In contrast to the FOO, the ROO estimates only those states that are not directly measured. An asymptotic observer in a lower dimension was studied in [14] for linear time-delay systems when the so-called observer matching condition is not satisfied. However, in most existing FOO/ROO works, the finite time convergence could not be guaranteed simultaneously with the invariance property for mismatched uncertain systems with a time-delay. The VSC problem has been investigated by researchers [4], [15], [16]. In [15], the state feedback controller was designed by using the average dwell time approach and the piecewise Lyapunov function technique. Based on concept of Moore–Penrose inverse, sliding mode variable structure control was studied for a class of mismatched uncertain switched delay systems [16]. In [4], the Lyapunov-Krasovskii function method and Leibniz-Newton formula were adopted to analyze the stability problems for uncertain time-delay systems, which did not consider perturbations. However, in some cases in physical control systems, the states of plant are not available for direct measurement or the sensor price is very expensive. Recently, many authors have presented several control schemes for the uncertain systems with time delay using only output variables [3], [5], [6], [17], [18]. In [17], FOO-based adaptive sliding mode controller for uncertain systems with a time delay under matching condition. H∞ non-fragile observer-based sliding mode controller was represented in [18] for uncertain time-delay systems subjected to input nonlinearity via linear matrix inequality (LMI) technique. This LMI technique [19] has some benefits over traditional approach methods; that is, LMI problems can be easily determined and efficiently solved by using the LMI Toolbox [20] in MATLAB software.

Manuscript published on 28 February 2017.
* Correspondence Author (s)
Yao-Wen Tsai, Mechanical and Automation Engineering, Da-Yeh University, Changhua 51591, Taiwan R.O.C, E-mail: ywtsai@mail.dyu.edu.tw, ywtsai.tw@gmail.com
Cong-Trang Nguyen, Mechanical and Automation Engineering, Da-Yeh University, Changhua 51591, Taiwan R.O.C, E-mail: congtrang0810@gmail.com

However, the error of state observer is asymptotically stable only, that is, the trajectories of closed-loop systems could not tend to zero in finite time. FOO-SMC was explored in [3] for a class of state-delayed switched systems with uncertain disturbance which is bounded by output signal. Recently, the study [5] was conducted to design SMC based on FOO for nonlinear Markovian jump systems with partly unknown transition probabilities. In [6], the SMC was proposed, which assumed that the norm of states and norm of observer error have to be bounded output signal, for a class of uncertain neural systems with unmeasurable states. It should be pointed out that the recent works [3], [5], [6] have some serious limitations, where it is required that the exogenous disturbances must be bounded by a known function of the outputs. Moreover, all published works have represented the SMC based on FOO, which increases the computation of burden due to the associated closed-loop systems.

Motivated by the aforementioned analysis, in this study, a class of time delay systems is considered involving both matched and mismatched uncertainties. The ROO is first designed to estimate unmeasured variables with lower-dimensional systems. Then, a finite time sliding mode controller (FTSMC) is designed by using only output variables and estimated variables. This FTSMC will drive the trajectory of the system to the switching surface in finite time and maintains a sliding mode. Finally, constructing LMI condition to guarantee the time-delay systems with mismatched uncertainties in sliding mode is asymptotically stable. Finally, the design procedure is given to summarize the proposed theoretical results.

The remainder of this paper is organized as follows: system model and preliminaries are considered in this paper is described in Section 2. It is also shown that the main results will be represented in Section 3. Stability analysis in sliding mode is showed in Section 4. Finally, a conclusion is provided in Section 5.

Notation: Throughout this paper, a variable/vector with subscript $d(t)$ is introduced to denote the time-delayed variable/vector, for example x_d denotes $x_d(t) = x(t-d)$. R^n symbolizes the n-dimensional Euclidean space, and $R^{n\times m}$ denotes the set of all $n\times m$ real matrices. For matrix A, the notation $A>0$ means that the matrix A is a positive definite matrix. I and 0 represent the identity matrix and a zero matrix, respectively. The superscript "T" shows the transpose. Finally, the notation $\|\bullet\|$ stands for the Euclidean norm of a vector and the induced spectral norm of a matrix.

II. SYSTEM MODEL AND PRELIMINARIES

We consider a class of the mismatched uncertain systems represented by the following equations

$$\begin{aligned}\dot{x}(t) &= [A+\Delta A(t)]x(t)+[A_d+\Delta A_d]x_d \\ &\quad + B[u(t)+\xi(x(t),x_d,t)], \\ y(t) &= Cx(t), \\ x_d &= x(t-d(t)) \text{ and } x(t)=\phi(t) \text{ for } -\bar{d}\le t<0,\end{aligned} \quad (1)$$

where the state variables, the control input, and the output of the system are represented by $x(t)\in R^n$, $u(t)\in R^m$, and $y(t)\in R^p$, respectively. The function $d(t)$ is a known non-negative constant time-delay and $\phi(t)$ is the continuous initial function defined over $[-\bar{d}, 0]$. The matrices A, A_d, B, and C are non-unique constant matrices with appropriate dimensions. The matrices $\Delta A(t)$ and ΔA_d represent the structure parameter mismatched uncertainties in the state matrix and the delayed state matrix, respectively. The term $\xi(x(t),x_d,t)$ describes the influence of exogenous disturbance on the plant.

Before proceeding with the main result of this paper, the several standard assumptions are needed for our study as follows

Assumption 1. The number of inputs is smaller than or equal to the number of output channels, that is, $m\le p<n$. The input matrix B and C have full rank, and $\text{rank}(CB)=m$.

Assumption 2. The pair $(A+A_d, B)$ is completely controllable, and the pair $(A+A_d, C)$ is completely observable.

Assumption 3. $\Delta A(t)$ and ΔA_d are mismatched parameter uncertainties which are assumed to be the form of $[\Delta A(t)\ \Delta A_d] = [D\Sigma(x(t),x_d,t)E\ \ D_d\Sigma_d(x(t),x_d,t)E_d]$, where D, E, D_d, and E_d constant matrices with appropriate dimensions, and $\Sigma(x(t),x_d,t)$ and $\Sigma_d(x(t),x_d,t)$ are unknown matrix function satisfying $\|\Sigma(x(t),x_d,t)\|\le 1$ and $\|\Sigma_d(x(t),x_d,t)\|\le 1$ for all $t\ge 0$, respectively.

Assumption 4. The unknown disturbance $\xi(x(t),x_d,t)$ of system (1) satisfies $\|\xi(x(t),x_d,t)\| \le k_\xi + k_m(\|x(t)\|+\|x_d\|)$, where k_ξ, and k_m are known non-negative constants.

Remark 1. Assumptions 1, 2, and 3 are standard assumptions for time delay systems which can be found in most existing literatures. For Assumption 4, in recent studies [3], [5], [6], it is required in this technical note that the exogenous disturbances must be bounded by a known function of the outputs. In practical cases, these conditions are often difficult to meet. For our method, the external disturbances must be satisfied an unknown function of the state and delayed state variables. Thus, the condition in Assumption 4 is an extension of the condition used in these studies.

To design a novel FTSMC for the uncertain time-delay system (1), the following sliding surface is chosen

$$\sigma(t) = Fy(t) = FCx(t) = Sx(t) = 0, \quad (2)$$

where $F\in R^{m\times p}$ is a constant matrix, and $S\in R^{m\times n}$ is a sliding matrix. It follows from (2), one can see that there are only output variables used.

For development of theorems and stability analysis in sliding mode, some following standard lemmas are necessary

Lemma 1 (see [21]). Let R_1, R_2 and $\Sigma(t)$ are real matrices of suitable dimension with $\Sigma^T\Sigma\le I$ then, for any scalar $\varphi>0$, the following matrix inequality holds

Published By:
Blue Eyes Intelligence Engineering
and Sciences Publication (BEIESP)

$$R_1\Sigma(t)R_2 + R_2^T\Sigma^T(t)R_1^T \le \varphi^{-1}R_1R_1^T + \varphi R_2^T R_2.$$
(3)

Lemma 2 (see [22]). For two vectors x, y of R^n and a positive definite matrix $N \in R^{nxn}$, following inequality holds

$$x^T Ny + y^T Nx \le \upsilon^{-1}x^T Nx + \upsilon y^T Ny,$$
(4)

for all $\upsilon > 0$.

Lemma 3 (see [19]). For a given matrix $\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{bmatrix}$ with $\Delta_{11}^T = \Delta_{11}$ and $\Delta_{22}^T = \Delta_{22}$, then the following conditions are equivalent

(i) $\Delta < 0$,
(ii) $\Delta_{11} < 0,\ \Delta_{22} - \Delta_{12}^T\Delta_{11}^{-1}\Delta_{12} < 0$,
(iii) $\Delta_{22} < 0,\ \Delta_{11} - \Delta_{12}^T\Delta_{22}^{-1}\Delta_{12} < 0.$
(5)

III. DESIGN OF THE OBSERVER-FINITE TIME SLIDING MODE CONTROLLER

A. *Description of a Regular Form*

First of all, an original time-delay system (1) is transformed into a regular form via a nonsingular transformation. According to the research of El-Ghezawi et al. [23], Zak and Hui [24]**.** There exist the eigenvector matrix $W \in R^{nx(n-m)}$ and the feedback matrix $N \in R^{mxn}$ such that $[A + BN]W = WJ$, where $J \in R^{(n-m)x(n-m)}$ is a freely chosen Jordan matrix which determines the system dynamics restricted to the switching surface. Using the procedures of the well-known eigenvalue/eigenvector assignment method, the following assumption is necessary

Assumption 5. The matrix $T \triangleq [W \quad B]$ is invertible and the inverse of T has the form $T^{-1} \triangleq \begin{bmatrix} W^g \\ B^g \end{bmatrix}$, where W^g and B^g denote the generalized inverse of W and B, respectively. By selecting $S = B^g$ and using the fact $TT^{-1} = I_n$, one can obtain

$$SB = I_m,\quad W^gW = I_{n-m},$$
$$SW = 0_{mx(n-m)},\quad W^gB = 0_{(n-m)xm}.$$
(6)

To obtain the regular form of the system (1), we introduce a new coordinate transformation as

$$\begin{bmatrix} z(t) \\ \sigma(t) \end{bmatrix} = T^{-1}x(t),\ \text{and}\ \begin{bmatrix} z_d \\ \sigma_d \end{bmatrix} = T^{-1}x_d,$$
(7)

where $z(t) \in R^{n-m}$ is unmeasurable variable, whereas the switching variable $\sigma(t) \in R^m$ is measurable.
Then transform the original system (1) into the regular form through the new coordinates (7) as

$$\begin{bmatrix} \dot z \\ \dot\sigma \end{bmatrix} = \begin{bmatrix} W^g\left[A+\Delta A(t)\right]W & W^g\left[A+\Delta A(t)\right]B \\ S\left[A+\Delta A(t)\right]W & S\left[A+\Delta A(t)\right]B \end{bmatrix}\begin{bmatrix} z \\ \sigma \end{bmatrix} + \begin{bmatrix} W^g\left[A_d+\Delta A_d\right]W & W^g\left[A_d+\Delta A_d\right]B \\ S\left[A_d+\Delta A_d\right]W & S\left[A_d+\Delta A_d\right]B \end{bmatrix}\begin{bmatrix} z_d \\ \sigma_d \end{bmatrix} + \begin{bmatrix} W^gB \\ SB \end{bmatrix}u(t) + \begin{bmatrix} W^gB \\ SB \end{bmatrix}\xi(x(t), x_d, t)$$
(8)

According to (6), equation (8) can be represented as

$$\begin{aligned} \dot z &= W^g\left[A+\Delta A(t)\right]Wz + W^g\left[A+\Delta A(t)\right]B\sigma \\ &\quad + W^g\left[A_d+\Delta A_d\right]Wz_d + W^g\left[A_d+\Delta A_d\right]B\sigma_d \\ \dot\sigma &= S\left[A+\Delta A(t)\right]Wz + S\left[A+\Delta A(t)\right]B\sigma \\ &\quad + S\left[A_d+\Delta A_d\right]Wz_d + S\left[A_d+\Delta A_d\right]B\sigma_d \\ &\quad + u(t) + \xi(x(t), x_d, t), \end{aligned}$$
(9)

where $z = W^gx,\ \sigma = Sx,\ z_d = W^gx_d$, and $\sigma_d = Sx_d$.
The equation (9) is rewritten

$$\begin{aligned} \dot z &= \left[J + W^g\Delta A(t)W\right]z + W^g\left[A+\Delta A(t)\right]B\sigma \\ &\quad + \left[J_d + W^g\Delta A_dW\right]z_d + W^g\left[A_d+\Delta A_d\right]B\sigma_d, \\ \dot\sigma &= S\left[A+\Delta A(t)\right]Wz + S\left[A+\Delta A(t)\right]B\sigma \\ &\quad + S\left[A_d+\Delta A_d\right]Wz_d + S\left[A_d+\Delta A_d\right]B\sigma_d \\ &\quad + u(t) + \xi(x(t), x_d, t), \end{aligned}$$
(10)

where $J = \bar A = W^gAW,\ J_d = \bar A_d = W^gA_dW.$

B. *Design of the Finite Time Sliding Mode Controller*

In this section, we will have three tasks. Firstly, the suitable ROO is constructed to estimate the unmeasurable states. Secondly, a FTSMC is designed by utilizing the estimated states and measured output. Finally, the design algorithm is also established to satisfy the reachability condition, that is,

$$\sigma^T(t)\dot\sigma(t) \le -\alpha\|\sigma(t)\|,$$
(11)

where α is positive scalar; $\sigma(t) = Fy(t)$ is the sliding function.

In order to estimate unmeasurable states, we propose suitable reduced-order observer for uncertain time-delay system (1) as follows

$$\dot{\hat z}(t) = J\hat z(t) + J_d\hat z_d + W^gAB\sigma(t) + W^gA_dB\sigma_d,$$
(12)

Published By:
Blue Eyes Intelligence Engineering
and Sciences Publication (BEIESP)

where $\hat{z}(t)=\hat{\phi}_z(t)=W^g\hat{\phi}(t)$ with $t\in[-\bar{d},0]$. Let the estimate of $z(t)$ and z_d be denoted by $\hat{z}(t)$ and $\hat{z}_d$, respectively. An error difference between the estimate state and the true state be defined by $e(t)$, i.e., $e(t)=\hat{z}(t)-z(t)$. Then, using the first equation (10) and equation (12) lead to the time delay observer error as

$$\dot{e}(t)=Je(t)+J_de_d-W^g\Delta A(t)Wz(t)-W^g\Delta AB\sigma(t) -W^g\Delta A_dWz_d-W^g\Delta A_dB\sigma_d,$$
(13)

where $e(t)=\phi_e(t)=W^g\left(\hat{\phi}(t)-\phi(t)\right),\ -d\le t\le 0.$

Now, we divide the problem of controller design into two cases including matched and mismatched uncertainties, respectively. The corresponding results will be considered in the following cases.

Case 1. Design the FTSMC for the matched uncertain time-delay system

If the uncertain terms $\Delta A(t)$ and ΔA_d satisfy the matching condition. The regular form of the system (1) can be represented as

$$\dot{z}(t)=Jz(t)+W^gAB\sigma(t)+J_dz_d+W^gA_dB\sigma_d,$$
(14)

$$\dot{\sigma}(t)=SAWz(t)+SAB\sigma(t)+SA_dWz_d+SA_dB\sigma_d + u(t)+\xi(x(t),x_d,t),$$
(15)

where $\xi(x,t)$ represents the lumped uncertainties and/or nonlinearities.

The error dynamic equation (13) can be reduced as follows

$$\dot{e}(t)=Je(t)+J_de_d,$$
(16)

Referring to [25], one can see that the error dynamic (16) is asymptotically stable if there exist symmetric matrices $K_1>0$ and $K_2>0$ such that the following LMI satisfies

$$\begin{bmatrix} JK_1+K_1J^T+K_2 & K_1J_d^{-1} \\ J_d^TK_1 & -K_2 \end{bmatrix}<0$$
(17)

Remark 2. As a sliding mode, $\sigma(t)=\sigma_d=0$, incurs in the system (14), we obtain $\dot{z}(t)=Jz(t)+J_dz_d$. Thus, the uncertain system (14) is asymptotically stable in sliding mode when it satisfies the LMI (17).

In order to use the estimated variables and observer error in controller design, we establish the following Lemma 4.

Lemma 4. Let error $e(0)$ be an initial condition of the error $e(t)$. The norm of estimation error $\|e(t)\|$ is bounded by $\eta_1(t)$ for all time. The term $\eta_1(t)$ is the solution of

$$\dot{\eta}_1(t)=\left[\lambda_{\max}+k\beta_1\|J_d\|\right]\eta_1(t),$$
(18)

where $\eta_1(0)\ge k\|e(0)\|$, $\lambda=\lambda_{\max}+k\beta_1\|J_d\|<0$ and $\lambda_{\max}$ is the maximum eigenvalue of J.

Proof of Lemma 4. By solving (16) to yields

$$\begin{aligned}\|e(t)\| &\le \|\exp(Jt)\|\|e(0)\|+\int_0^t\left\|\exp\left[J(t-\tau)\right]\right\|\|J_de_d\|d\tau, \\ &\le k\exp(\lambda_{\max}t)\|e(0)\|+\int_0^t k\exp\left[\lambda_{\max}(t-\tau)\right] \\ &\quad\times\|J_d\|\|e_d\|d\tau\end{aligned}$$
(19)

For the inequality (19), we multiply both sides by the term $\exp(-\lambda_{\max}t)$, then

$$\|e\|\exp(-\lambda_{\max}t)\le k\|e(0)\|+\int_0^t k\exp(-\lambda_{\max}\tau) \times\|J_d\|\|e_d\|d\tau$$
(20)

Based on Lemma 3 of [26], we have $\|e_d\|\le\beta_1\|e(t)\|$ for some scalar $\beta_1>1$. Let $\exp(-\lambda_{\max}t)$ be the right-hand side term of the inequality (20), we attain

$$\begin{aligned}\|e(t)\| &\le k\|e(0)\|\exp\left[\left(\lambda_{\max}+k\beta_1\|J_d\|\right)t\right], \\ &\le \eta(0)\exp\left[\left(\lambda_{\max}+k\beta_1\|J_d\|\right)t\right] \\ &=\eta_1(t)\end{aligned}$$
(21)

Thus, $\|e(t)\|$ is bounded by $\eta_1(t)$ for all time. □

Now, we design control input $u(t)$ in system (15), the control input will be appropriately designed with the help of the ROO tool (12).

Theorem 1. The matched uncertain system (14), (15) under the output feedback controller

$$\begin{aligned}u(t)&=u_1(t) \\ &=-k_1\sigma(t)-\left\{k_2\left[\|\hat{z}(t)\|+\eta_1(t)\right]+k_3+k_4\|\sigma_d\|\right\}\frac{\sigma(t)}{\|\sigma(t)\|}\end{aligned} \quad (22)$$

reach the switching surface σ(t) = 0 in finite time and stay on it thereafter if the constant gains satisfy the following conditions

Published By:
Blue Eyes Intelligence Engineering
and Sciences Publication (BEIESP)

$$k_1 > k_m \|B\| + \|SAB\|,$$
$$k_2 > \|SAW\| + \varepsilon_1 \|SA_d W\| + k_m \|W\| (1+\varepsilon_1),$$
$$k_3 > k_\xi + \alpha,$$
$$k_4 > k_m \|B\| + \|SA_d B\|,$$
(23)

where $\alpha > 0$, $\eta_1(t)$ is defined in Lemma 4, and k_1, k_2, k_3, k_4 are constant gains.

Proof of Theorem 1. From state transformation (7) implies that

$$x(t) = Wz(t) + B\sigma(t),$$
$$x_d = Wz_d + B\sigma_d$$
(24)

Since $z(t) = \hat{z}(t) - e(t)$, $z_d = \hat{z}_d - e_d$, and $\|e(t)\| \le \eta_1(t)$. In addition, it follows from the reference [26] that for any solution $x(t+d(t))$ of (1), there exists a constant $\varepsilon_1 > 1$ such that $\|z_d\| \le \varepsilon_1 \|z(t)\|$. Therefore, the equation (24) can be rewritten as

$$\|x(t)\| \le \|W\| \left[\|\hat{z}\| + \eta_1(t) \right] + \|B\| \|\sigma(t)\|,$$
$$\|x_d\| \le \|W\| \|\hat{z}_d\| + \|B\| \|\sigma_d\|.$$
(25)

We consider the Lyapunov function $V(\sigma(t)) = 0.5\sigma^T(t)\sigma(t)$. It follows from (25), Assumption 4, and the equation of the system (15) that

$$\begin{aligned}\sigma^T(t)\dot{\sigma}(t) &= \sigma^T \left[SAWz(t) + SAB\sigma(t) + SA_d Wz_d + SA_d B\sigma_d + u(t) + \xi(x(t), x_d, t) \right] \\ &\le \|\sigma(t)\| \Big\{ \left(\|SAW\| + \varepsilon_1 \|SA_d W\| \right) \left[\|\hat{z}(t)\| + \eta_1(t) \right] + \|SA_d B\| \|\sigma_d\| + \|SAB\| \|\sigma(t)\| \Big\} + \sigma^T(t) u_1(t) \\ &\quad + \|\sigma(t)\| \Big\{ k_\xi + k_m \left[\left(\|W\| + \varepsilon_1 \|W\| \right) \left(\|\hat{z}(t)\| + \eta_1(t) \right) \right] + \|B\| \left(\|\sigma(t)\| + \|\sigma_d\| \right) \Big\}.\end{aligned} \quad \textbf{(26)}$$

Then, using the controller in (22) and (23) implies that the reachability condition (11) is required. Accordingly, $\sigma(t) = 0$ will be reached within finite time by Lyapunov stability theory. The proof is ended. □

Case 2. Design the FTSMC for the mismatched uncertain time-delay system

In this case, we will extend the design techniques advanced in *Case 1* for the mismatched uncertain system with a time delay. Assume that the uncertain terms $\Delta A(t)$ and ΔA_d do not satisfy the matching condition. Generally, the time delay observer error dynamics can be abridged as (13).

Lemma 5. Consider the error dynamic of equation (13). Let λ_{max} be the maximum eigenvalue of J. Then, the following two statements hold:

(a) $\|\exp(Jt)\| \le k \exp(\lambda_{\max} t)$ for some $k > 0$.

(b) $\|e(t)\|$ is bounded by $\eta_2(t)$ for all time, where $\eta_2(t)$ is the solution of

$$\begin{aligned}\dot{\eta}_2(t) &= \lambda \eta_2(t) + k\gamma \left(\|W^g D\| \|EW\| + \beta_2 \|W^g D_d\| \|E_d W\| \right) \\ &\quad + k \left(\|W^g D\| \|EB\| \|\sigma(t)\| + \|W^g D_d\| \|E_d B\| \|\sigma_d\| \right),\end{aligned} \quad \textbf{(27)}$$

where γ is constant, $\lambda = \lambda_{\max} + k\beta_2 \|J_d\| < 0$, and $\eta_2(0) \ge k\|e(0)\| > 0$.

Proof of Lemma 5. Since all the eigenvalues of J are real negative, therefore $\lambda_{\max} < 0$, and (a) can be obtained immediately. Now, we will mainly prove (b). Solving (13) to yield

$$\begin{aligned}\|e\| &\le \|e^{Jt}\| \|e(0)\| + \int_0^t \|e^{J(t-\tau)}\| \Big\| J_d e_d + W^g \Delta AWz + W^g \Delta AB\sigma + W^g \Delta A_d Wz_d + W^g \Delta A_d B\sigma_d \Big\| d\tau, \\ &\le k \exp(\lambda_{\max} t) \|e(0)\| + \int_0^t k \exp\left[\lambda_{\max}(t-\tau) \right] \Big\| J_d e_d + W^g \Delta AWz + W^g \Delta AB\sigma + W^g \Delta A_d Wz_d + W^g \Delta A_d B\sigma_d \Big\| d\tau.\end{aligned} \quad \textbf{(28)}$$

Multiply the term $\exp(-\lambda_{\max} t)$ to both sides of (28) gives

$$\begin{aligned}\|e\| \exp(-\lambda_{\max} t) &\le k \|e(0)\| + \int_0^t k \exp\left[\lambda_{\max}(t-\tau) \right] \Big(\|J_d\| \|e_d\| + \|W^g D\| \|EW\| \|z\| + \|W^g D\| \|EB\| \|\sigma\| + \|W^g D_d\| \\ &\quad \times \|E_d W\| \|z_d\| + \|W^g D_d\| \|E_d B\| \|\sigma_d\| \Big) d\tau.\end{aligned}$$

(29)

Based on Lemma 3 of [26], we have $\|z_d\| \le \beta_2 \|z(t)\|$ for some scalar $\beta_2 > 1$. In addition, assume that there is a positive constant γ such that $\|z(t)\| \le \gamma$. The inequality (29) can be rewritten as

$$\begin{aligned}\|e(t)\| \exp(-\lambda_{\max} t) &\le k \|e(0)\| + \int_0^t k \exp\left[\lambda_{\max}(t-\tau) \right] \\ &\quad \times \Big(\|J_d\| \|e_d\| + \left(\|W^g D\| \|EW\| + \beta_2 \|W^g D_d\| \|E_d W\| \right) \gamma + \|W^g D\| \|EB\| \|\sigma(t)\| + \|W^g D_d\| \|E_d B\| \|\sigma_d\| \Big) d\tau.\end{aligned} \quad \textbf{(30)}$$

Shift the term $\exp(-\lambda_{\max} t)$ to the right-hand side of inequality (30), then

Published By:
Blue Eyes Intelligence Engineering
and Sciences Publication (BEIESP)

$$
\begin{aligned}
\|e(t)\| \le \eta_2(0)\exp\left[\left(\lambda_{\max} + k\beta_2\|J_d\|\right)t\right] \\
+ \int_0^t k\beta_2 \exp\left(\lambda_{\max} + k\beta_2\|J_d\|\right)(t-\tau) \\
\times\left[\left(\|W^g D\|\|EW\| + \beta_2\|W^g D_d\|\|E_d W\|\right)\gamma + \|W^g D\| \right.\\
\left.\times\|EB\|\|\sigma\| + \|W^g D_d\|\|E_d B\|\|\sigma_d\|\right]d\tau \\
= \eta_2(t),
\end{aligned}
$$

(31)

where $\eta_2(t)$ satisfies (27). Therefore, Lemma 5 is proved. □

In the next step, the control law is continuously designed to make the system remain on the sliding surface, which based on the help of ROO tool (12) and the result of Lemma 5. Now, assume that the FTSMC is of the following form

$$
\begin{aligned}
u(t) &= u_2(t) \\
&= -\bar{k}_1\sigma(t) - \left[\bar{k}_2\left(\|\hat{z}(t)\| + \eta_2(t)\right) + \bar{k}_3 + \bar{k}_4\|\sigma_d\|\right]\frac{\sigma(t)}{\|\sigma(t)\|},
\end{aligned}
\quad \textbf{(32)}
$$

where $\eta_2(t)$ is defined in Lemma 5, α is positive scalar, and $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$, $\bar{k}_4$ are constant gains and will be designed later.

Theorem 2. Consider the mismatched uncertain system with a time delay (9) subject to all assumptions. If the error dynamic (13) satisfies Lemma 5 and the switching matrix *F* satisfies the equation *S* = *FC*, then under the control law (32), the mismatched uncertain system (9) is driven to the switching surface *Fy* = 0 in finite time and maintains the sliding mode when the constant gains satisfy the following conditions

$$
\begin{aligned}
&\bar{k}_1 > \|SAB\| + \|SD\|\|EB\| + k_m\|B\|, \\
&\bar{k}_2 > \left(\|SAW\| + \varepsilon_2\|SA_d W\|\right) + (\varepsilon_2+1)\|SD_d\|\|E_d W\| \\
&\qquad + k_m(\varepsilon_2+1)\|W\|, \\
&\bar{k}_3 > k_\xi + \alpha, \\
&\bar{k}_4 > \|SA_d B\| + \|SD_d\|\|E_d B\| + k_m\|B\|.
\end{aligned}
$$

(33)

Proof of Theorem 2. The Lyapunov function candidate is selected as $V(\sigma(t)) = 0.5\sigma^T(t)\sigma(t)$. Then, differentiating $V(\sigma(t))$ with regard to time and using the second equation (9), we have

$$
\begin{aligned}
\dot{V}(\sigma(t)) = \sigma^T(t)\{SAWz(t) + S\Delta A(t)Wz(t) + SA_d Wz_d \\
+ S\Delta A_d Wz_d + SAB\sigma + S\Delta A(t)B\sigma + SA_d B\sigma_d \\
+ S\Delta A_d B\sigma_d + u_2(t) + \xi(x(t), x_d, t)\}.
\end{aligned}
\quad \textbf{(34)}
$$

Similarly, with the state transformation T in (7), $z(t) = \hat{z}(t) - e(t)$, $z_d = \hat{z}_d - e_d$, $\|e\| \le \eta_2$, and $\|z_d\| \le \varepsilon_2\|z(t)\|$ for some scalar $\varepsilon_2 > 1$ implies that

$$
\begin{aligned}
&\|x(t)\| \le \|W\|\left(\|\hat{z}(t)\| + \eta_2(t)\right) + \|B\|\|\sigma(t)\|, \\
&\|x_d\| \le \varepsilon_2\|W\|\left(\|\hat{z}(t)\| + \eta_2(t)\right) + \|B\|\|\sigma_d\|.
\end{aligned}
$$

(35)

Substituting the controller (32), gains (33), and (35) into the equality (34), we obtain

$$
\begin{aligned}
\dot{V}(\sigma(t)) \le \|\sigma(t)\|\{\left(\|SAW\| + \|SD\|\|EW\|\right)\|z(t)\| + \left(\|SA_d W\| \right.\\
\left. + \|SD_d\|\|E_d W\|\right)\|z_d\| + \left(\|SAB\| + \|SD\|\|EB\|\right)\|\sigma(t)\| \\
+ \left(\|SA_d B\| + \|SD_d\|\|E_d B\|\right)\|\sigma_d\|\} \\
+ \left(\sigma^T u_2(t) + \|\sigma(t)\|\xi(x(t), x_d, t)\right). \\
\le -\alpha\|\sigma\|.
\end{aligned}
$$

(36)

Thus, if $\sigma(t) \ne 0$, the reachability condition (11) is satisfied, which means that the sliding mode can be maintained by the FTSMC (32). This completes the proof of Theorem 2.

C. Establish the Procedures for Controller

Now, we summarize the all steps for the FTSMC in two cases that including matched and mismatched uncertain systems with a time delay as

Step 1. By selecting $S = B^g$ and computing the sliding matrix F such that equation $FC = S$ is satisfied.

Step 2. Replacing the matrix F into equation (2), the sliding function $\sigma(t)$ is found.

Step 3. The ROO $\hat{z}(t)$ is designed as equation (12).

Step 4. The FTSMCs are synthesised as follows:

For case 1 (*matched uncertainty*): Determine the upper bound of observer dynamic error $\eta_1(t)$ as (18). Then, design of the FTSMC $u_1(t)$ according to equation (22).

For case 2 (*mismatched uncertainty*): Determine the upper bound of observer dynamic error $\eta_2(t)$ as (27). Then, design of the FTSMC $u_2(t)$ according to equation (32).

IV. STABILITY ANALYSIS IN SLIDING MODE

Now, we are in the position to derive sufficient conditions in terms of LMI such that the first equation (10) is an asymptotically stable. Let us begin with considering the following LMI:

$$
\begin{bmatrix}
\Pi & \bar{E}^T & \bar{E}_d^T & H\bar{D} & H\bar{D}_d \\
\bar{E} & -\varphi^{-1}I & 0 & 0 & 0 \\
\bar{E}_d & 0 & -\left(\mu_2\varphi_{1d}\right)^{-1}I & 0 & 0 \\
\bar{D}^T H & 0 & 0 & -\varphi^{-1}I & 0 \\
\bar{D}_d^T H & 0 & 0 & 0 & -\varphi_{1d}^{-1}I
\end{bmatrix} < 0, \quad \textbf{(37)}
$$

where $\Pi = \bar{A}^T H + H\bar{A} + \varphi_{2d}^{-1}H + \mu_1\varphi_{2d}\bar{A}_d^T H\bar{A}_d$, $H \in R^{(n-m)x(n-m)}$ is any positive matrix, and the scalars $\varphi > 0$, $\varphi_{1d} > 0$, $\varphi_{2d} > 0$, $\mu_1 > 1$, and $\mu_2 > 1$. Then, we can establish the following Theorem 3.

Published By:
Blue Eyes Intelligence Engineering and Sciences Publication (BEIESP)

Theorem 3. Suppose that LMI (37) has a solution $H > 0$, the scalars $\varphi > 0$, $\varphi_{1d} > 0$, $\varphi_{2d} > 0$, $\mu_1 > 1$, and $\mu_2 > 1$. The sliding function is given by equation $\sigma(t) = Fy(t)$. Then, the first equation of the closed-loop system (10) exhibits an asymptotically stable sliding mode.

Proof of Theorem 3. In the sliding mode, we have $\sigma(t) = 0$ and $\dot{\sigma}(t) = 0$. The first equation of (10) can be rewritten

$$\dot{z}(t) = \left[\bar{A} + \bar{D}\Sigma\bar{E}\right] z(t) + \left[\bar{A}_d + \bar{D}_d\Sigma_d\bar{E}_d\right] z_d, \quad \textbf{(38)}$$

where $\bar{A} = J = W^g AW$, $\bar{A}_d = J_d = W^g A_d W$, $\bar{D} = W^g D$, $\bar{D}_d = W^g D_d$, $\bar{E} = EW$, and $\bar{E}_d = E_d W$.

To analyze the stability of the sliding motion (38), we consider the following Lyapunov function candidate

$$V(t) = z^T(t) P z(t) \quad \textbf{(39)}$$

Then, taking the time derivative along the state trajectory of (39), we have

$$\begin{aligned}\dot{V}(t) = {} & z^T(t)\left[\bar{A}^T H + H\bar{A} + \bar{E}^T\Sigma^T\bar{D}^T H + H\bar{D}\Sigma\bar{E}\right] z(t) \\ & + z_d^T \bar{E}_d^T \Sigma_d^T \bar{D}_d^T H z(t) + z^T H \bar{D}_d^T \Sigma_d \bar{E}_d^T z_d \\ & + z^T(t) H \bar{A}_d z_d + z_d^T \bar{A}_d^T H z(t)\end{aligned} \quad \textbf{(40)}$$

Now, we are going to prove $\dot{V}(t) < 0$. By using Lemma 1, it follows from equation (40) that

$$\begin{aligned}\dot{V}(t) \le {} & z^T(t)\left[\bar{A}^T H + H\bar{A} + \varphi^{-1} H\bar{D}\bar{D}^T H + \varphi\bar{E}^T\bar{E}\right] z(t) \\ & + \varphi_{1d}^{-1} z^T(t) H\bar{D}_d\bar{D}_d^T H z(t) + \varphi_{1d} z_d^T \bar{E}_d^T \bar{E}_d z_d \\ & + z^T(t) H\bar{A}_d z_d + z_d^T \bar{A}_d^T H z(t).\end{aligned} \quad \textbf{(41)}$$

By virtue of Lemma 2, inequality (41) is equivalent to

$$\begin{aligned}\dot{V}(t) \le {} & z^T(t)\left[\bar{A}^T H + H\bar{A} + \varphi^{-1} H\bar{D}\bar{D}^T H + \varphi\bar{E}^T\bar{E} + \varphi_{2d}^{-1} H \right. \\ & \left. + \varphi_{1d}^{-1} H\bar{D}_d\bar{D}_d^T H\right] z + z_d^T\left[\varphi_{1d}\bar{E}_d^T\bar{E}_d + \varphi_{2d}\bar{A}_d^T H\bar{A}_d\right] z_d.\end{aligned} \quad \textbf{(42)}$$

According to Assumption 3, E is a free-choice matrix. So, we can easily select matrix E such that the matrix $E^T E$ is semi-positive definite. Then, from the Lemma 3 of [26], the following is true

$$z_d^T \bar{A}_d^T H\bar{A}_d z_d \le \mu_1 z^T \bar{A}_d^T H\bar{A}_d z \quad \textbf{(43)}$$

for some $\mu_1 > 1$, which implies that

$$z_d^T \bar{E}_d^T \bar{E}_d z_d \le \mu_2 z^T \bar{E}_d^T \bar{E}_d z, \quad \textbf{(44)}$$

where the scalar $\mu_2 > 1$. Thus, from inequality (42), (43), and (44), we achieve

$$\begin{aligned}\dot{V}(t) \le {} & z^T\left[\bar{A}^T H + H\bar{A} + \varphi_{2d}^{-1} H + \mu_1\varphi_{2d}\bar{A}_d^T H\bar{A}_d + \varphi\bar{E}^T\bar{E} \right. \\ & \left. + \varphi^{-1} H\bar{D}\bar{D}^T H + \mu_2\varphi_{1d}\bar{E}_d^T\bar{E}_d + \varphi_{1d}^{-1} H\bar{D}_d\bar{D}_d^T H\right] z.\end{aligned} \quad \textbf{(45)}$$

Then, by the application of Lemma 3 to LMI (37), we have

$$\begin{aligned}& \Pi + \varphi\bar{E}^T\bar{E} + \mu_2\varphi_{1d}\bar{E}_d^T\bar{E}_d + \varphi^{-1} H\bar{D}\bar{D}^T H \\ & + \varphi_{1d}^{-1} H\bar{D}_d\bar{D}_d^T H < 0.\end{aligned} \quad \textbf{(46)}$$

After all, from equation (45) and (46), it is obvious that

$$\dot{V}(t) < 0. \quad \textbf{(47)}$$

The inequality (47) implies that if LMI (37) is feasible, the system (10) in the sliding mode is asymptotically stable.

V. CONCLUSION

This paper deals with the VSC design problem for a class of matched and mismatched uncertain systems with a time delay. The suitable reduced-order observer has been constructed, which ensures that the conservatism is reduced, and the robustness is enhanced in comparison with FOO. The FTSMC for reaching motion has been designed such that the trajectories of closed-loop system can be moved onto the sliding surface and maintained there in finite time. Finally, by employing the Lyapunov stability theory and LMIs technology, the resulting sliding mode dynamics is asymptotically stable under sufficient condition established.

ACKNOWLEDGMENT

The authors would like to acknowledge the financial support provided by the National Science Council in Taiwan (NSC 102-2632-E-212-001-MY3).

REFERENCES

1. J . Y. Hung, W. Gao, and J. C. Hung, "Variable structure control: A survey," IEEE Trans. on Indus. Electro., vol. 40, 1993, pp. 2–22.
2. S. H. Zak, Systems and Control, New York: Oxford Univ. Press, 2003.
3. Z. He, X. Wang, Z. Gao, and J. Bai, "Sliding mode control based on observer for a class of state-delayed switched systems with uncertain perturbation," Mathematical Problems in Engineering, vol. 2013, Article ID 614878, 2013, 9 pages.
4. S.-H. Tsai, and Y.-H. Chang,"Sliding Mode Control for A Class of Uncertain Time-Delay System,"Ap. Mathematics and Infor. Sci., 2015, pp. 267-273.
5. H. Li, P. Shi, D. Yao, and L. Wu, "Observer-based adaptive sliding mode control for nonlinear Markovian jump systems," Automatica, vol. 64, Feb. 2016, pp. 133–142.
6. Z. Liu, L. Zhao, Y. Kao, and C. Gao, "Robust passive control for a class of uncertain neutral systems based on sliding mode observer," ISA Transactions, http://dx.doi.org/10.1016/j.isatra.2016.10.013, 2016.
7. Boulaabi, A. Sellami, and F.B. Hmida, "A robust sensor fault reconstruction based on a new sliding mode observer design method for uncertain delayed systems: A satellite system application," J. of Con. Engineering and Ap. Informatics, vol. 17, 2015, pp. 88–97.
8. C.T. Chen and S. T. Peng, "A sliding mode control scheme for non-minimum phase non-linear uncertain input-delay chemical processes," J. of Process Control, vol. 16, 2006, pp. 37–51.
9. Rezoug, B. Tondu, M. Hamerlain, and M. Tadjine, "RBFNN-HOMS Non-singular Terminal Sliding Control of n-DOF Robotic Manipulator," J. of Control Engineering and Ap. Infor., vol. 18, 2016, pp. 52–62.

10. S. C. Garcia, M. C. M. Teixeira, J. P. F. Garcia, U. N. L. T. Alves, and J. M. S. Ribeiro, "Discrete-Time Sliding Mode Control for Uncertain Networked System Subject to Time Delay," Mathematical Problems in Engineering, vol. 2015, Article ID 195120, 2015, 10 pages.
11. Q. Zhang, "Adaptive observers for multiple–input–multiple–output (MIMO) linear time-varying systems," IEEE Trans. Autom. Control, vol. 47, Mar. 2002, pp. 525–529.
12. J. Koshkouei, K. J. Burnham,"Discontinuous Observers for Nonlinear Time-Delay Systems,"Int. J. Systems Science, vol. 40, 2009.
13. F. J. Bejarano and A. Pisano, "Switched observers for switched linear systems with unknown inputs," IEEE Trans. Automat. Control, vol. 56, Mar. 2011, pp. 681–686.
14. F. Zhu, "State estimation and unknown input reconstruction via both reduced-order and high-order sliding mode observers," J. Process Control, vol. 22, 2012, pp. 296-302.
15. L. Wu and J. Lam, "Sliding mode control of switched hybrid systems with time-varying delay," Int. J. Adapt. Control Signal Process., vol. 22, 2008, pp. 909–931.
16. J. Lian, J. Zhao, and G. M. Dimirovski, "Robust H∞ sliding mode control for a class of uncertain switched delay systems," Int. J. Syst. Sci., vol. 40, 2009, pp. 855–866.
17. M. Pai,"Observer-based adaptive sliding mode control for nonlinear uncertain state-delayed systems,"International J. of Control, Auto., and Sys., vol. 7, 2009, pp. 536-544.
18. L. Liu, Z. Han, and W. Li, "H∞ non-fragile observer-based sliding mode control for uncertain time-delay systems," J. Franklin Inst., vol. 347, 2010, pp. 567–576.
19. S. Boyd, L. El. Ghaoui, E. Feron, and V. Balakrishnan, Linear Matrix Inequalities in Systems and Control Theory, Philadelphia, PA: SIAM, 1994.
20. P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, LMI Control Toolbox, Natick, MA: The MathWorks, Inc, 1994.
21. Y. W. Tsai and V. V. Huynh, "Output Feedback and Single-Phase Sliding Mode Control for Complex Interconnected Systems," Mathematical Problems in Engineering, vol. 2013, Article ID 946385, 2015, 16 pages.
22. S. Sayyaddelshad and T. Gustafsson, "H∞ observer design for uncertain nonlinear discrete-time systems with time-delay: LMI optimization approach," Inter. J. of Robust and Nonlinear Control, vol. 25, 2015, pp. 1514–1527.
23. O. M. E. El-Ghezawi, A. S. I. Zinober, and S. A. Billings, "Analysis and design of variable structure systems using a geometric approach," International J. of Control, vol 38, 1983, pp. 657-671.
24. S. H. Zak and S. Hui, "On variable structure output feedback controllers," IEEE Trans. Autom. Control, vol. 38, 1993, pp. 1509–1512.
25. K. Gu, V. L. Kharitonov, and J. Chen, Stability of Time-delay Systems, Birkh¨auser: Boston, 2003.
26. X. G. Yan, S. K. Spurgeon, and C. Edwards, "Static output feedback sliding mode control for time-varying delay systems with time-delayed nonlinear disturbances," Inter. J. Robust Nonlinear Control, vol. 20, 2010, pp. 777–788.

Published By:
Blue Eyes Intelligence Engineering
and Sciences Publication (BEIESP)