The Scaling-Free CORDIC Using Generalized Micro-Rotation Selection

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Abstract—This paper presents an CORDIC algorithm that completely eliminates the scale-factor. By suitable selection of the order of approximation of Taylor series the proposed CORDIC circuit meets the accuracy requirement, and attains the desired range of convergence. Besides we have proposed an algorithm to redefine the elementary angles for reducing the number of CORDIC iterations. A generalized micro-rotation selection technique based on high speed most-significant-1-detection obviates the complex search algorithms for identifying the micro-rotations. The proposed CORDIC processor provides the flexibility to manipulate the number of iterations depending on the accuracy, area and latency requirements.

Keywords- CORDIC.

I. INTRODUTION

The coordinate rotation digital computer (CORDIC) has established its popularity in several important areas of application, like generation of sine and cosine functions, calculation of discrete sinusoidal transforms like fast Fourier transform (FFT), discrete sine/cosine transforms (DST/DCT), householder transform (HT), etc. [1]–[3].

Many variations have been suggested for efficient implementation of CORDIC with less number of iterations over the conventional CORDIC algorithm [4]–[11]. The number of CORDIC iterations are optimized in [4]–[6] by greedy search at the cost of additional area and time for the implementation of variable scale-factor. In [7] and [8] efficient scale-factor compensation techniques are proposed which adversely affect the latency/throughput of computation. Two area-time efficient CORDIC architectures have been suggested in [9], which involve constant scale-factor multiplication for adequate range of convergence (RoC).

The virtually scale-free CORDIC in [10] also requires multiplication by constant scale-factor and relatively more area to achieve respectable RoC. The enhanced scale-free CORDIC in [11] combines few conventional CORDIC iterations with scaling-free CORDIC iterations for an efficient pipelined CORDIC implementation with improved RoC. In this paper, we propose a novel scaling-free CORDIC algorithm for area-time efficient implementation of CORDIC with adequate RoC. The proposed recursive architecture has comparable or less area complexity with other existing scaling-free CORDIC algorithms. Moreover, no scale-factor multiplications are required for extending the RoC to entire coordinate space.

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II. BRIEF OVERVIEW OF CORDIC ALGORITHM

The CORDIC algorithm operates either in, rotation mode or vectoring mode, following linear, circular or hyperbolic coordinate trajectories. In this paper, we focus on rotation mode CORDIC using circular trajectory.

A. Conventional CORDIC Algorithm

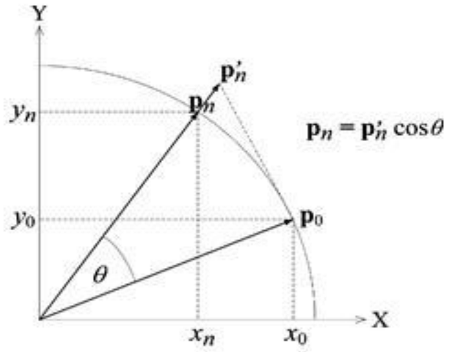


Fig 1 : Rotation Of A Vector On A Two-Dimensional Plane.

As shown in Fig.1, the rotation of a Two-dimensional vector $P_0=[X_0 , Y_0]$ through an angle θ to obtain a rotated vector Pn= [Xn, Yn] could be performed by the matrix product **Pn=R.P$_0$** where

R is the rotation matrix:

$$\mathbf{R}= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{.................. (1)}$$

By factoring out the cosine term in (1), the rotation matrix **R** can be rewritten as

$$R= [(1+\tan^2\theta)^{-1/2}] \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \quad \text{..................(2)}$$

this can be interpreted as a product of a scale-factor $K=[(1+\tan^2\theta)^{-1/2}]$with a pseudo rotation matrix Rc, given by

$$RC= \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \quad \text{...................................(3)}$$

The pseudo rotation operation rotates the vector P_0 by an angle θ and changes its magnitude by a factor K=cosθ to produce a pseudo-rotated vector p'n = R_c P_0. To achieve simplicity of hardware realization of the rotation, the key ideas used in CORDIC arithmetic are to (i) decompose the rotations into a sequence of elementary rotations through predefined angles that could be implemented with minimum hardware cost; and (ii) to avoid scaling, that might involve arithmetic operation, such as square-root and division. The second idea is based on the fact that scale-factor contains only the magnitude information but no information about the angle of rotation.

Iterative Decomposition Of the Angle Of Rotation

The CORDIC algorithm performs the rotation iteratively by breaking down the angle of rotation into a set of small pre defined angles, $\alpha i = \tan^{-1}(2^{-i})$, so that $\tan \alpha i = 2^{-i}$ could be implemented in hardware by shifting through bit locations. Instead of performing the rotation directly through an angle θ, CORDIC performs it by a certain number of micro rotations through angle α_i where

$$\theta = \sum_{i=0}^{b} \mu_i \cdot \alpha_i$$

$$\alpha_i = \tan^{-1}(2^{-i}).$$

σ_i OR $\mu_i \in \{1, -1\}$

This satisfies the CORDIC convergence theorem . Note that the range of convergence of this algorithm is limited to [99.99°, 99.99°], which means the angular decomposition of (4) is applicable for angles in the first and fourth quadrants. This can be extended to entire coordinate space using the properties of sine and cosine functions, using an extra iteration for full-range rotation. where the rotation matrix for the *i*th iteration corresponding to the selected angle αi is given by

$$\mathbf{R}(i) = K_i \begin{bmatrix} 1 & -\sigma_i 2^{-i} \\ \sigma_i 2^{-i} & 1 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots.(5)$$

The scale-factor, and the pseudo rotation matrix are given by

$$\mathbf{R_c}(i) = \begin{bmatrix} 1 & -\sigma_i 2^{-i} \\ \sigma_i 2^{-i} & 1 \end{bmatrix}$$

$$K_i = 1/\sqrt{(1+2^{-2i})} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Note that the pseudo-rotation matrix $Rc(i)$ for the i^{th} iteration alters the magnitude of the rotated vector by a scale-factor K_i during the i^{th} micro rotation, which is independent of the value of σi (direction of micro rotation) used in the angle decomposition.

Avoidance Of Scaling

Scaling-free CORDIC was the first attempt to completely dispose of the scale-factor by Voider which completely removes scale factor Ki from (5) The removal of scaling from the iterative micro rotations leads to a pseudo-rotated vector $P'n=R_cP_0$ instead of the desired rotated vector $P_n=K R_c P_0$, where the scale-factor K is given by

$$K = \prod_{i=0}^{n} K_i = \prod_{i=0}^{n} 1/\sqrt{(1+2^{-2i})}. \quad \ldots\ldots\ldots\ldots\ldots.(7)$$

Since the scale-factor of micro rotations does not depend on the direction of micro rotations and decreases monotonically, the final scale-factor K converges to 0.60725.Therefore, instead of scaling during each micro rotation, the magnitude of final output could be scaled by K. Here, the sine and cosine functions were approximated to

$$\sin \alpha_i = 2^{-i} \qquad \cos \alpha_i = 1 - 2^{-(2i+1)}.$$

However, the approximation imposes a restriction on the basic-shift i = [(b-2.585)/3] (The minimum possible permissible shifts in the CORDIC iteration have been termed as basic shift, which is equal to the number of right shifts in the first CORDIC iteration). For 16-bit data, the basic-shift=4 results in extremely low range of convergence. However, *modified virtually adaptive scaling-free* algorithm [10], extends the range of convergence over the entire coordinate space and introduces an adaptive scale-factor.

B. Proposed Algorithm For Scaling Free CORDIC

The proposed design is based on the following key ideas: 1) use of Taylor series expansion of sine and cosine functions to avoid scaling operation and 2) suggest a generalized sequence of micro-rotation to have adequate range of convergence (RoC) based on the chosen order of approximation of the Taylor series.

Taylor Series Approximation of Sine and Cosine Functions

The Taylor expansions of sine and cosine of an angle "α" are given by

$$\sin\alpha = \alpha - (3!)^{-1}\cdot\alpha^3 + (5!)^{-1}\cdot\alpha^5 \ldots$$
$$\cos\alpha = 1 - (2!)^{-1}\cdot\alpha^2 + (4!)^{-1}\cdot\alpha^4 \ldots.$$

Estimating for the maximum error in the evaluation of sine and cosine functions for different Order of approximations. The maximum percentage of error in sine and cosine functions for third order approximation is 0.0033% and 0.0168%, respectively, within the permissible CORDIC elementary angles range of [7π/88]. Therefore, I choose third order of approximation for Taylor's expansion of sine and cosine functions.

Representation of Micro-Rotations Using Taylor Series Approximation

Using different orders of approximation of sine and cosine functions in (2), we can have

$$x_{i+1} = \left(1 - \frac{\alpha_i^2}{2!}\right)\cdot x_i - \left(\alpha_i - \frac{\alpha_i^3}{3!}\right)\cdot y_i$$
$$y_{i+1} = \left(1 - \frac{\alpha_i^2}{2!}\right)\cdot y_i + \left(\alpha_i - \frac{\alpha_i^3}{3!}\right)\cdot x_i$$
$$x_{i+1} = \left(1 - \frac{\alpha_i^2}{2!} + \frac{\alpha_i^4}{4!}\right)\cdot x_i - \left(\alpha_i - \frac{\alpha_i^3}{3!}\right)\cdot y_i$$
$$y_{i+1} = \left(1 - \frac{\alpha_i^2}{2!} + \frac{\alpha_i^4}{4!}\right)\cdot y_i + \left(\alpha_i - \frac{\alpha_i^3}{3!}\right)\cdot x_i$$
$$x_{i+1} = \left(1 - \frac{\alpha_i^2}{2!} + \frac{\alpha_i^4}{4!}\right)\cdot x_i - \left(\alpha_i - \frac{\alpha_i^3}{3!} + \frac{\alpha_i^5}{5!}\right)\cdot y_i$$
$$y_{i+1} = \left(1 - \frac{\alpha_i^2}{2!} + \frac{\alpha_i^4}{4!}\right)\cdot y_i + \left(\alpha_i - \frac{\alpha_i^3}{3!} + \frac{\alpha_i^5}{5!}\right)\cdot x_i$$
$$x_{i+1} = \left(1 - \frac{\alpha_i^2}{2!} + \frac{\alpha_i^4}{4!} - \frac{\alpha_i^6}{6!}\right)\cdot x_i - \left(\alpha_i - \frac{\alpha_i^3}{3!} + \frac{\alpha_i^5}{5!}\right)\cdot y$$
$$y_{i+1} = \left(1 - \frac{\alpha_i^2}{2!} + \frac{\alpha_i^4}{4!} - \frac{\alpha_i^6}{6!}\right)\cdot y_i + \left(\alpha_i - \frac{\alpha_i^3}{3!} + \frac{\alpha_i^5}{5!}\right)\cdot x$$

$$x_{i+1} = \left(1 - \frac{\alpha_i^2}{2!} + \frac{\alpha_i^4}{4!} - \frac{\alpha_i^6}{6!}\right)\cdot x_i - \left(\alpha_i - \frac{\alpha_i^3}{3!} + \frac{\alpha_i^5}{5!} - \frac{\alpha_i^7}{7!}\right)\cdot y_i$$
$$y_{i+1} = \left(1 - \frac{\alpha_i^2}{2!} + \frac{\alpha_i^4}{4!} - \frac{\alpha_i^6}{6!}\right)\cdot y_i + \left(\alpha_i - \frac{\alpha_i^3}{3!} + \frac{\alpha_i^5}{5!} - \frac{\alpha_i^7}{7!}\right)\cdot x_i.$$

Since the errors resulting from the five combinations are of very small order, preference is given for first pair of equations for coordinate calculation, which satisfies the accuracy and RoC requirements with minimum complexity.

Expressions for Micro-Rotations Using Taylor Series Approximation and Factorial Approximation

Although, we find that we can use Taylor series expansion with third order of approximation, with desired accuracy and RoC requirement, the above selected equation cannot be used in the CORDIC shift-add iterations. To implement the equation by shift-add operations, we need to approximate the factorial terms by the power of 2 values, replacing 3! by 2^3 in the equation we get

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} \left(1-(2!)^{-1}\cdot\alpha_i^2\right) & -\left(\alpha_i - 2^{-3}\cdot\alpha_i^3\right) \\ \left(\alpha_i - 2^{-3}\cdot\alpha_i^3\right) & \left(1-(2!)^{-1}\cdot\alpha_i^2\right) \end{bmatrix} \cdot \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

Determination of the Basic-Shift for a Given Order of Approximation of Taylor Series Expansion

One can find that: 1) the order of approximation of Taylor series expansion of sine and cosine functions determines the basic-shift to be used for CORDIC iterations, and 2) the basic-shift of CORDIC microoperation determines the range of convergence. The expressions for the basic-shifts, the first elementary angle of rotation (α_1) and RoC for different orders of approximations for different word-length of implementations are as follows:

$$\text{basic-shift},\ s = \left\lceil \frac{b - \log_2(n+1)!}{(n+1)} \right\rceil$$

$$\text{RoC} = n_1 \times \alpha_1, \quad \alpha_1 = 2^{-s}.$$

Where b is the word length and n_1 is number of micro rotations.

The values in Table I are derived from the above equations. It is seen that with increase in the order of approximation, the basic-shift decreases, the first elementary angle of rotation increases and RoC is expanded. Very often inclusion of higher order terms does not have any impact on the accuracy for smaller word-lengths. The basic-shift for third order of approximation using (7a), for 16-bit word-length is 2.854. The RoC (with basic-shift for 16-bit) is large enough to be mapped to the entire coordinate-space.

TABLE I
COMPARISON OF APPROXIMATION ORDERS VERSUS ROC FOR VARIOUS BIT WIDTHS BASED ON (7)

Order of Approx.	Basic-Shift		First Elementary Angle (Radians)		RoC for $n_1 = 4$ (Radians)	
	16-bit	32-bit	16-bit	32-bit	16-bit	32-bit
3	2	6	0.25	0.01562	1	0.0625
4	1	5	0.5	0.03125	2	0.125
5	1	3	0.5	0.125	2	0.5

C. Generalized Micro-Rotation Selection

In the proposed generalized micro-rotation sequence, we perform multiple iterations of basic-shift, followed by non-repetitive unidirectional iterations of the micro-rotations corresponding to other shift indices, to minimize the number of iterations and achieve adequate range of convergence.

Organization of Micro-Rotation Sequence

In the proposed scheme, the rotation angle "θ" can be represented as

$$\theta = n_1 \cdot \alpha_s + \sum_{i=1}^{n_2} \alpha_{s_i}, \quad n = n_1 + n_2$$

where α_s is the elementary angle corresponding to the basic-shift, α_{si} are elementary angles for other shifts, *n1* and *n2* are non-negative integers and *n* represents the total number of iterations. If we do not use any micro-rotation of angle α_s then *n1* is zero, and *n2=n*. On the other hand, if the desired angle of rotation "θ" is a multiple of α_s then *n2* is zero and *n1=n.*

Defining the Elementary Angles

The elementary angles α_s and $\alpha_{si}$are given by

$$\alpha_s = 2^{-s} \quad \text{and} \quad \alpha_{s_i} = 2^{-s_i} \qquad \ldots\ldots\ldots (9)$$

where, *s* is the basic-shift and $s_i > s$ is the shift for i^th^ iteration. For basic-shift = 2, we can find α_s=(7π/88) and for basic-shift =3 , we can find α_s=(7π/176) . In Table II, lists the decimal and (0, 16) fixed point binary representation of the elementary angles corresponding to different shifts.

Generalized Micro-Rotation Sequence Identification

The micro-rotations are identified depending on the bit representation of the desired rotation angle in radix-2 system using most-significant-1 detector. For this reason the maximum rotation angle is restricted to π/4 radians as the entire coordinate space [0,2π] can be mapped to the [0,π/4]using octant symmetry of sine and cosine functions If the most-significant-1 location (M) of the rotation angle "θ" is smaller than the basic-shift "s", elementary angle of the basic-shift would be used for the CORDIC iteration. For a fixed word-length of N-bit, the shift (s_i) for the elementary angle is given by

$$s_i = N - M$$

TABLE II
BIT REPRESENTATION OF ELEMENTARY ANGLES AND CORRESPONDING SHIFTS

Shift (s_i)	Elementary Angle (α_i)	
	Decimal	16-bit Hexadecimal
2	0.25	4000 H
3	0.125	2000 H
4	0.0625	1000 H
5	0.03125	0800 H

III. CORDIC ARCHITECTURE

The block diagram for the proposed CORDIC architecture is shown below.

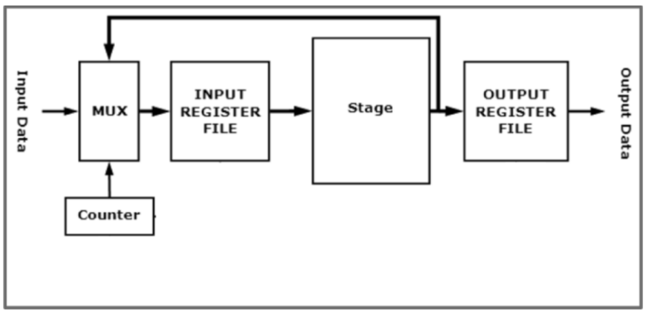


Fig. 2 Architecture of the proposed CORDIC processor.

The number of iterations required in a CORDIC processor decides the rollover count of the counter. The rollover count is seven for basicshift =2 and ten for basic-shift =3. The expiry of the counter signals the completion of a CORDIC operation; depending on this signal, the multiplexer either loads a new data-set (rotation angle, initial value of "x" and "y") to start a fresh CORDIC operation, or recycles the output of the stage to begin a new iteration for the current CORDIC operation. The input and output register files act as latches for synchronization.

A. The stage

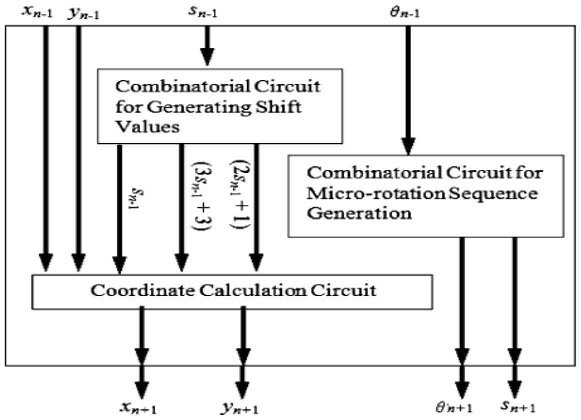
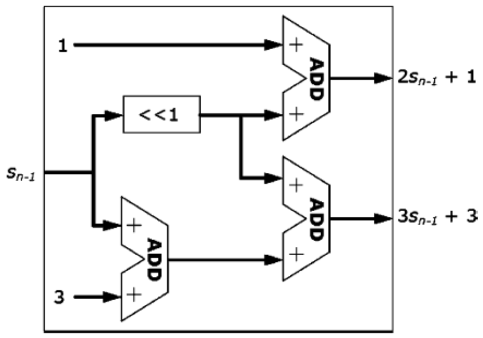


Fig 3 Block diagram for the stage.

It makes use of the same stage for all the iterations for the coordinate calculations, as well as for the generation of shift values. The structure of each stage (shown in Fig. 3) consists of three computing blocks namely: 1) shift-value estimation; 2) coordinate calculation; and 3) micro-rotation sequence generator.

B. The Generation of Shift values



C.

Fig 4. Combinational circuit for generating the shift values.

The combinational circuit for generating the shift values are explainable from the below equation

TABLE III
PSEUDO CODE FOR GENERATING THE MICRO-ROTATION SEQUENCE

Input: angle to be rotated (θ_i)
Begin
M = Most-Significant-1 Location of θ_i
  if (M == 15) then
    α = 0.25 radians
    Shift, s_i = 2 and $\theta_{i+1} = \theta_i - \alpha$
  Else
    Shift, s_i = 16 – M
    $\theta_{i+1} = \theta_i$ with θ_i[M] = '0'
End

D. The Micro Rotation of the angle

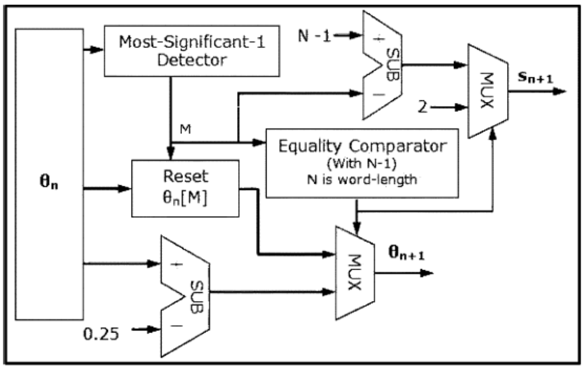


Fig 5. Combinational circuit of micro rotation sequence generation.

The micro rotation sequence generation involves the splitting of the angle to be rotated into smaller values. As seen from the fig 5, the angle to be rotated is converted into the hexadecimal number. The most significant-1 detector identifies which bit is logic high, starting from the MSB towards the LSB. If the detected bit is positioned >14 bit position, then the angle to be rotated will be subtracted from 0.25 degrees and the shift=2; if the bit position is <14 then angle to be rotated will be subtracted from 0.125 degrees and shift=3. The pseudo code for the same can be seen from the table III.

IV. IMPLEMENTATION

The Scaling free CORDIC Algorithm has been implemented in MATLAB 7.5. The vector rotation has been successfully achieved with the aid of micro rotation sequence generation..

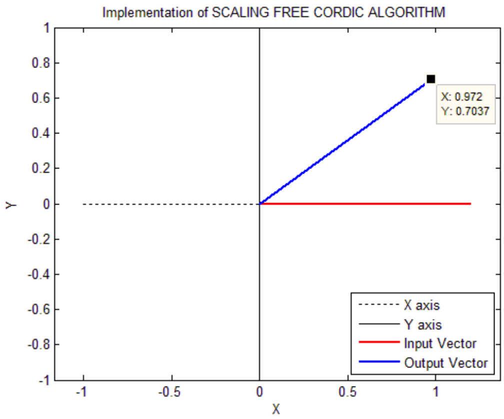


Fig 6: Implementation of the CORDIC Algorithm for vector rotation.

V. CONCLUSION

The proposed algorithm provides a scale-free solution for realizing vector-rotations using CORDIC. The order of Taylor series approximation is decided appropriately by the proposed algorithm, not only to meet the accuracy requirement but also to attain Adequate range of convergence. The Generalized micro-rotation selection technique is suggested to reduce the number of iterations for low latency implementation.
Moreover, a high speed most-significant-1 detection scheme obviates the complex search algorithms for identifying the micro-rotations.

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