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Theorems on Planar Graphs

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Abstract: Graph coloring is a well-known and well-studied area of graph theory with many applications. In this paper, we will discuss list precoloring extensions.

Key Words: 5-list-coloring, 2-connected, P -separating 3-cycle.

I. INTRODUCTION

Theorm: Note if the two precolored vertices are adjacent, then the coloring is extendable by Thomassen's 5-list-coloring theorem. In general, we use induction on |V (G)| where the base case is precolored u and v connected by an edge. Assume G is connected, otherwise the result follows trivially by induction.

Claim. G has no separating 3-cycle or 4-cycle.

Let U be a vertex set of such a separating cycle. By the assumption of the theorem, U does not separate {u,v}. Let V_1 and V_2 be the vertex sets of disconnected plane graphs obtained by removing G[U] from G, such that $\{u, v\} \subseteq V_i \cup U$. By induction, color G[$V_1 \cup U$] from L. This gives a proper coloring c of U . Now, in G[$V_2 \cup U$], there is a face with vertex set U having color lists of size 1 and all other vertices have color lists of size 5. Thus, by known Theorem , G[$V_2 \cup U$] is colorable from the corresponding lists.

Let S = $v_0\, v_1$ ……. V_m be a shortest (u, v)-path in G, with v_0 = u and v_m = v, for m ≥ 2. By

Lemma there is a nice coloring c of $v_0 v_1$ …… v_{m-2} . By known Lemma there is at most one vertex adjacent to v_{m-2}, v_{m-1}, v_m and at most one vertex adjacent to v_{m-3}, v_{m-2}, v_{m-1}, if m ≥ 3. Let c(v_{m-1}) L(v_{m-1}) - ({c(v_{m-2})} $\cup$ L(v_m)).

If there is no vertex x, with x~{v_{m-2}, v_{m-1}, v_m }, and no vertex x, with x~{v_{m-3}, v_{m-2},v_{m-1}}, then c is a nice coloring of S .

Assume that there is a vertex y, with y ~ {v_{m-3}, v_{m-2}, v_{m-1}}, and there is no vertex x, with x ~ {v_{m-2}, v_{m-1}, v_m }, or, the other way around, there is no vertex x, with x ~ {v_{m-3}, v_{m-2}, v_{m-1}}. and there is a vertex y, with y ~ {v_{m-2}, v_{m-1}, v_m }.Then c is a proper coloring of S such that |L_c(p)| ≥ 3 for every p N (S) - {y}, and |L_c(y)| ≥ 2. Deleting S and the corresponding colors from the lists of their neighbors in G - S produces a list assignment where all vertices in a face containing N (S) have lists of size at least 3 (except for y), and all other vertices have lists of size 5. Using Thomassen's 5-list-coloring theorem, G - S can be colored from these lists. Together with the coloring c of S, it gives a proper L-coloring of G.

Finally, assume there is a vertex x, with x ~ {v_{m-3}, v_{m-2}, v_{m-1} }, and there is a vertex w, with w ~ {v_{m-2}, v_{m-1}, v_m }. Note that there is at most one additional vertex adjacent to v_{m-1} and v_m, call it z if it exists. Delete S from G and add two new adjacent vertices t and s in the resulting face, also add edges xt, ws, tz, sz, ty_i, where y ∈N (v_{m-1}) and sx_i, where x_i ∈ N(v_m). Choose two new colors a and ß not used in any of the lists assigned to vertices of G. Let L' (t) = {α}, L'(s) = {β}, L'$_c$(y_i) = L_c (y_i) {α}, L (x_i) = L'$_c$(x_i) {β}, L'(z) = L_c(z) $\cup$ {α, β}, L'(x) = L_c (x) $\cup$ {α}, and L' (w) = L_c(w) $\cup$ {β}. For every other vertex of this modified graph, let L be equal to L'. See Figure 3.9 for an illustration of this process. Observe that L satisfies the conditions of Thomassen's 5-list-coloring theorem, so there is a proper L'- coloring of this graph. Thus, there is a proper L -coloring of G - S, where no vertex uses colors α or β. This is a proper L_c-coloring of G - S. Together with the coloring c of S, it gives a proper L-coloring of G.

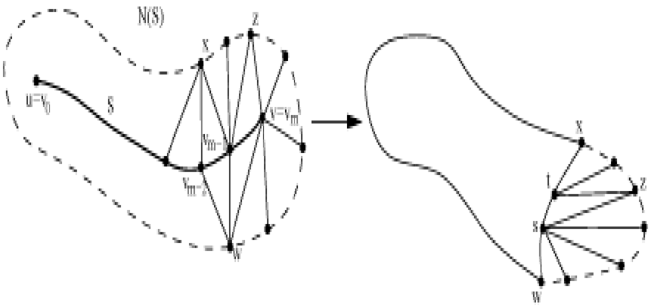

Figure 3.9: The addition of vertices t and s in G – S.

Theorem . Let T be a (P, 45)-Steiner tree in G", a Type I reduced graph of G satisfying the conditions of the theorem. Let L be an assignment of lists of colors to vertices of G such that |L(v)| = 1 for v P and |L(v)| = 5 for v P . We first color G, then extend it to a proper L-coloring of G.

To color G", first color special vertices of T which are not in P arbitrarily from their lists. Let S be the set of branches in T and let S ∈ S with endpoints u_S, v_S. Let H(S, u_S, v_S) = H(S) be the graph obtained by applying Lemma 3.20 to S and c be a nice coloring of H(S) from the corresponding lists (see Figure 3.10). Finally, let c be a coloring of H = $\cup_{S \in S}$ H(S), such that c(v) = c_S (v) if v ∈ H(S).

Claim 1. The coloring c is a nice coloring of H.

Let x, x' be two vertices of H that do not belong to the same H(S). We shall prove that x and x' do not have common neighbors outside of H and they are not adjacent. Let x ∈ H(S), x' ∈H(S'), S, S' ∈ S, S ≠ S .

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If x, x' $\in$ V (T), then x and x do not have a common neighbor outside of T and they are not adjacent by part (4) of the definition of a (P, d)-Steiner tree.

If x $\in$ V (T), x' $\notin$ V (T), then x' V (H(S')) - V (S'), thus dist(x', $v_{C'}$) $\leq$ 21, where $v_{C'}$ is a center of S , as follows from Lemma 3.20. From part (3) of the definition of a (P, d)-Steiner tree, we have that dist($v_{C'}$, x) $\geq$ d. Thus dist(x, x') $\geq$ d - 21 $\geq$ 3 when d $\geq$ 24.

Finally if x, x' $\notin$ V (T), then x $\notin$ V (H(S)) - V (S) and x' V (H(S')) - V (S'). Thus dist(x, v_C) $\leq$ 42 and dist(x', v_c) $\leq$ 42, where v_c, v_c are centers of S and S , respectively. Moreover dist(v_c, v_c) $\geq$ d. Thus dist(x, x') $\geq$ d - 42 $\geq$ 3 if d $\geq$ 45.

It follows that c is a proper coloring of H. To show that c is nice, consider a vertex v adjacent to H. We see that v is adjacent to non-special vertices of H(S) for at most one branch S of T . Since c is a nice coloring of H(S), it follows that $|L_c(v)| \geq 3$.

To conclude the proof of Claim 1, recall that H is a connected graph containing all vertices of P . By known Proposition , implies that G" is L-colorable. To show that G is L-colorable, it is sufficient to observe the following.

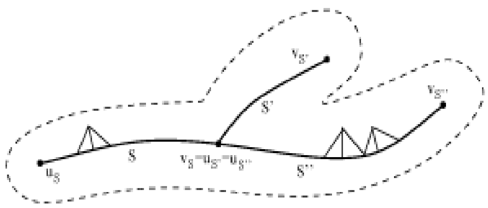

Figure 3.10: An example of the graph H obtained in the proof of Theorem 3.4.

Claim 2. Let F be a graph, P be a set of vertices, and L be an assignment of lists of size 5 to vertices of V (G) - P and lists of size 1 to vertices of P. Let F' = R(F) be a reduction of F .

If F has a proper coloring from lists L then F has a proper coloring from lists L.

Let c be a proper coloring of F from lists L.

If F' was obtained from F by removing the vertices in a region separated by 3-cycle or 4-cycle, these vertices can be colored properly from L using above theorem.

If F was obtained from F by removing the set X of 4 vertices, y_1, y_2, z_1, z_2 of configuration D, we see that $|L_c(y_i)| \geq 2$, i = 1, 2 for the two vertices y_1, y_2 of degree two in F[X] and $|L_c(z_i)| \geq 3$, i = 1, 2, for the two vertices z_1, z_2 of degree three in F [X]. In the subgraph F [X] each vertex has list size equal to its degree under list assignment L_c. An L_c -coloring of F [X] can be found directly or by the results . Thus F has a proper coloring from lists L. If F was obtained from F by removing the set X of 7 vertices w, x_1, . . ., x_6 of configuration W , then we see that $|L_c(x_1)|$, $|L_c(x_4)| \geq 2$, $|L_c(x_2)|$, $|L_c(x_3)|$, $|L_c(x_5)|$, $|L_c(x_6)| \geq 3$, and $|L_c(w)| = 5$. Let $\alpha \in L_c(w) - (L_c(x_1) \bigcup L_c(x_4))$, so color w with α and remove α from $L_c(x_2)$, $L_c(x_3)$, $L_c(x_5)$, $L_c(x_6)$. What remains to be colored is a 6-cycle with vertices having lists of size at least 2, which is colorable by the classification of all 2-list-colorable graphs by Erdos et al. Since F[X] is properly colorable from lists L_c, F is properly colorable from lists L.

This proves Claim 2.

Since G" was obtained from G via a sequence of reductions, the theorem follows.

Theorem (1): Let L be an assignment of lists of colors to vertices of G such that |L(x)| = 5 for all x $\notin$ P and $|L(v_i)| = 1$ for all $v_i \in$ P. If P is a set of vertices and edges with pairwise distance at least 3, then for all x $\notin$ P, x is adjacent to at most two vertices of P. Thus, for every proper coloring c of G[P] from the corresponding lists L and for all x $\in$ P, we have $|L_c(x, P)| \geq 3$. Moreover, N (P) belongs to the frontier of a face in G - P . Thus, by Proposition 3.15 , G is colorable from lists L.

(2) Without loss of generality, assume C is on the unbounded face of G. Let P = {v_0, v_1, . . . , v_{k-1} } C be a set of at most six precolored vertices on the boundary of C . Fix an assignment L of lists of colors to the vertices of G with |L(v)| = 5 for all v $\in$ V (G) – P and $|L(v_i)| = 1$ for all v_i P. We shall show that G is L-colorable provided the three forbidden configurations are not present.

We shall create a new graph G' on the vertex set of G with new lists L. Let c_0, ….., c_{k-1} be distinct colors not present in L(v) for any v$\in$ V (G). Let L be a new list assignment with L' (v_i) := {c_i} for i = 0, .. . , k - 1 and L(v) = L(v) – S_v S'_v for each v $\in$ V (G) - P, where S_v is the set of colors used in lists L of vertices in P $\bigcap$ n N(v) and S'_v is an arbitrary subset of the set of colors used in lists L of vertices of P $\bigcap$ n N(v), such that $|S_v| = |S_v|$. In creating L we simply replaced the colors originally assigned to P with new distinct colors, and replaced the old colors in the lists of vertices in the neighborhood of vertices of P.

Let a new plane graph G be obtained from G by removing the edges v_i v_{i+1} for i = 0, . . . , k-1 that correspond to non-consecutive vertices of C, and adding all edges v_i v_{i+1} for i = 0, . . . , k -1 in the unbounded face of G. The resulting graph has a new unbounded face with vertex set P, and, perhaps, some new edges. By Theorem 7, G is L -colorable by a coloring c provided the three forbidden configurations are not present. Moreover, for any v $\notin$ P , we have c(v) $\notin$ {c_0, . . . , c_k } $\bigcup$ S_v, so c(v) L(v) and c(v) $\notin$ L(v_i) if v ~ v_i . To create a proper L-coloring of G, replace the color c_i with an element of L(v) for i=0, ..., k-1.

Theorem: Delete P and the corresponding colors from the lists of adjacent vertices. There are at most two faces, F'_1or F_1 and F_2, in the graph G -P such that the vertices adjacent to P in G belong to the boundaries of these two faces. These vertices have lists of size at least 4, and all other vertices in G - P have lists of size at least 5. Call the resulting lists L'. Add a vertex v_i to the face F'_i and make it adjacent to all vertices on F'_i, i = 1, or i = 1, 2. Let a be a color not used in any of the lists L(v), v $\in$ V . Let L" (v_1) = L (v_2) = {α}, L" (v) = L' (v) $\bigcup$ {α}, if v $\in$V (F'_1 F'_2) and |L(v)| = 4. For all other vertices, let L (v) = L (v). Applying Theorem 3.5 to the resulting graph with lists L allows for this graph to be properly colored from these lists. We note here that it is not hard to see that this new graph does not contain any {v_1,v_2 }-separating 3-cycles or 4-cycles because such a separating 3-cycle or 4-cycle would have to be made up of vertices and edges from the original graph and would have separated some of the precolored vertices of G, a contradiction. This coloring gives a proper coloring of G - P from lists L', and thus it gives a proper coloring of G from lists L.

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II. CONCLUSIONS

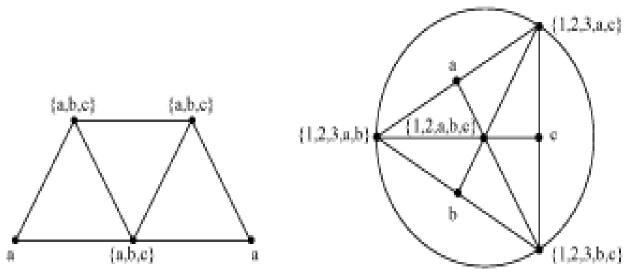
We proved the question of Albertson has a positive answer if there are no short cycles separating precolored vertices and there is a nice tree containing precolored vertices. We note here that by the definition of a (P, d)-Steiner tree, Theorem 3.4 can be applied to plane graphs with precolored vertices that are not far apart. For example, let G be a 100-cycle with vertices v_0, v_1, ….., v_{99} and P = {v_1, v_{50}, v_{98}}. Then G contains a (P, 48)-Steiner tree obtained from deleting v_0,v_{99} and incident edges. The centers of the branches are far apart, but dist(v_1, v_{98}) = 3.

We believe that in a planar triangulation either such a tree could always be found, or there are small reducible configurations such as shown in Figure 2.4. The reducible configurations D and W are just two in a family of many reducible configurations of those types, see Chapter 5. Modifying the definition of a reduced graph to include the removal of every reducible K_4 – e and every reducible 6 - wheel leads us to the following question.

Question: Is it the case that every reduced planar triangulation with a set P of precolored vertices with dist(P) ≥ 1000 contains a (P, 45)-Steiner tree?

If the above question has a positive answer, then by Theorem 3.4, the precoloring of P extends to a 5-list-coloring of G. We did not strive to improve the constants here. With more careful calculations, one could easily obtain smaller constants.

The condition of no separating short cycles seems to be essential. It is important to note that the condition in Thomassen's 5-list-coloring theorem that the two precolored vertices must be adjacent is essential. See Figure 3.11a . Also, the distance condition in this problem cannot be eliminated, even for a small number of precolored vertices. See Figure 3.11b. However, we conjecture that a precoloring of two far-apart vertices is always extendable to a 5-list-coloring of a planar graph.



(a) Non-extendable precoloring of two vertices at distance 2 where other vertices have lists of size 3.

(b) Non-extendable precoloring of three vertices at distance 2.

Figure 3.11: Non-extendable precolorings.

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